

ON FENCHEL-NIELSEN COORDINATES OF SURFACE GROUP REPRESENTATIONS INTO $SU(3,1)$

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ABSTRACT. Let Σ_g be a compact, orientable surface of genus $g \geq 2$. We ask the question of parametrizing discrete, faithful, totally loxodromic representations in the deformation space $Hom(\pi_1(\Sigma_g), SU(3,1))/SU(3,1)$. We show that such a representation, under some hypothesis, can be specified by $30g - 30$ real parameters.

1. INTRODUCTION

Let Σ_g be a closed orientable surface of genus $g \geq 2$ and let $\pi_1(\Sigma_g)$ be its fundamental group. The classical Teichmüller space can be considered as the space of discrete, faithful, totally loxodromic representations of $\pi_1(\Sigma_g)$ into $SL(2, \mathbb{R})$ up to conjugacy. To construct the Fenchel-Nielsen coordinates of the classical Teichmüller space, one starts by specifying a curve system of $2g - 2$ simple closed curves on Σ_g . The complement of such curve system decomposes the surface into $2g - 2$ three-holed spheres. A three-holed sphere is also known as a pair of pants in the literature. The Fenchel-Nielsen coordinates provides the degrees of freedom that is needed to glue these several pairs of pants to construct a hyperbolic surface. Given a discrete, faithful, totally loxodromic representation $\rho : \pi_1(\Sigma_g) \rightarrow SL(2, \mathbb{R})$, a pair of pants in the above pant-decomposition of the surface corresponds to a two-generator subgroup $\langle A, B \rangle$ generated by loxodromics such that AB is also loxodromic. The loxodromic elements $A, B, B^{-1}A^{-1}$ correspond to the boundary components of the pair of pants. Such a group is called a $(0, 3)$ group in the literature. It follows from a classical work of Fricke [4] and Vogt [24] that a $(0, 3)$ group in $SL(2, \mathbb{R})$ is completely determined by the traces of their generators and their product. For an up to date exposition of this work, see Goldman [6]. The gluing of the pairs of pants corresponds to gluing of these $(0, 3)$ groups. In the gluing process there are traces of these loxodromics, along with rotation of the peripheral or the boundary curves during the gluing. These rotation angles are called twist-bend parameters. The traces of the loxodromics along with the twist-bend parameters determine a representation $\rho : \pi_1(\Sigma_g) \rightarrow SL(2, \mathbb{R})$ completely up to conjugation.

Let $\mathbf{H}_{\mathbb{C}}^n$ be the n dimensional complex hyperbolic space. The group $SU(n, 1)$ acts as the holomorphic isometry group of $\mathbf{H}_{\mathbb{C}}^n$. A discrete, faithful, geometrically finite and totally loxodromic representation of a surface group into $SU(n, 1)$ is called a *complex hyperbolic quasi-Fuchsian representation*. The geometry of these representations is mostly unknown. In the last two decades, there have been some understanding of these representations when the target group is $SU(2, 1)$ and a conjectural picture of the representation space has been evolved, see Parker-Platis [15] and Schwartz [18] for surveys. However, when the target group is $SU(n, 1)$, $n \geq 3$, not much is known. Foundational information like classification of isometries of $\mathbf{H}_{\mathbb{C}}^n$ and their relationship with conjugacy invariants have also been obtained very recently, see [10, 8].

Parker and Platis [15] generalized the Fenchel-Nielsen coordinates for discrete, faithful, geometrically finite and totally loxodromic representations of $\pi_1(\Sigma_g)$ into the group $SU(2, 1)$. As a starting

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point of their Fenchel-Nielsen coordinate system, Parker and Platis [15, Theorem 7.1] proved a generalization of the result of Fricke-Vogt for two-generator Zariski-dense free subgroups of $SU(2, 1)$ with loxodromic generators. Parker and Platis followed an approach that uses traces of the generators and a point on the cross-ratio variety. In another approach, it follows from the work of Lawton [13], Wen [21] and Will [22, 23] that a two-generator free Zariski dense subgroup of $SU(2, 1)$ is determined by traces of the generators and the traces of three more compositions of the generators. For a survey of these results see Parker [14].

There have been generalization of Fenchel-Nielsen coordinates in three dimensional real hyperbolic geometry and projective geometry as well. Tan [20] and Kourouniotis [12] constructed Fenchel-Nielsen coordinates for quasi-Fuchsian representations of $\pi_1(\Sigma_g)$ into $SL(2, \mathbb{C})$. Goldman [7] generalized Fenchel-Nielsen coordinates on space of convex real projective structures on Σ_g . Recently, Strubel [19] has developed Fenchel-Nielsen coordinates for representations of $\pi_1(\Sigma_g)$ into $Sp(2n, \mathbb{R})$ with maximal Toledo invariant.

In this work we intend to generalize the work of Parker and Platis [15] for representations of $\pi_1(\Sigma_g)$ into $SU(3, 1)$. The reason for choosing $\mathbf{H}_{\mathbb{C}}^3$ is recent work of the authors [9, 10] where foundational information about isometries of $\mathbf{H}_{\mathbb{C}}^3$ has been obtained. The starting point, as in the classical case, is to parametrize $(0, 3)$ subgroups of $SU(3, 1)$. However for two-generator subgroups in $SU(3, 1)$ traces and cross-ratios of the generators are not sufficient to determine the subgroup up to conjugacy. For the determination of $(0, 3)$ subgroups in $SU(3, 1)$, one needs to look for more conjugacy invariants of the pair of generators. For this purpose, we use new invariants which are generalizations of Goldman's eta invariants. As we shall see, for 'generic' representations that we are calling *tame representations*, our invariants fit together nicely and it gives Fenchel-Nielsen type coordinates to specify such representation up to conjugacy.

Let $\mathbb{C}^{3,1}$ be the vector space \mathbb{C}^4 equipped with a non-degenerate Hermitian form of signature $(3, 1)$. Then $\mathbf{H}_{\mathbb{C}}^3$ is the projectivization of negative vectors in $\mathbb{C}^{3,1}$. The boundary $\partial\mathbf{H}_{\mathbb{C}}^3$ is the projectivization of null vectors. Following Goldman [5] recall that a k -dimensional complex totally geodesic subspace of $\mathbf{H}_{\mathbb{C}}^3$ or a \mathbb{C}^k -plane is the projectivization of a copy of $\mathbb{C}^{k,1}$ in $\mathbb{C}^{3,1}$, $k = 1, 2$. A \mathbb{C}^1 -plane is simply called a *complex geodesic*. A \mathbb{C}^k -chain is the boundary of a \mathbb{C}^k -plane in $\mathbf{H}_{\mathbb{C}}^3$; a \mathbb{C}^1 -chain is simply called a *chain*. A positive vector c is polar to a \mathbb{C}^2 -plane C if the lift of C in $\mathbb{C}^{3,1}$ is the orthogonal complement of c . The positive vector c is polar to a \mathbb{C}^2 -chain L if L is the boundary of a \mathbb{C}^2 -plane C that is polar to c .

For four distinct points z_1, z_2, z_3 and z_4 in $\partial\mathbf{H}_{\mathbb{C}}^3$ the *Koranyi-Riemann cross-ratio* is defined by:

$$(1.1) \quad \mathbb{X}(z_1, z_2, z_3, z_4) = \frac{\langle \mathbf{z}_3, \mathbf{z}_1 \rangle \langle \mathbf{z}_4, \mathbf{z}_2 \rangle}{\langle \mathbf{z}_4, \mathbf{z}_1 \rangle \langle \mathbf{z}_3, \mathbf{z}_2 \rangle},$$

where \mathbf{z}_i is lift of z_i in $\mathbb{C}^{3,1}$. For more details on cross ratios, see [5]. We extend the above definition to define invariants for the "generic case" that includes three null vectors and one positive vector in $\mathbb{C}^{3,1}$. For a loxodromic element A , we denote by $\mathbf{a}_A, \mathbf{r}_A$ the null eigenvectors of A corresponding to the fixed points and let \mathbf{x}_A and \mathbf{y}_A correspond to the positive eigenvectors of A .

Let A, B be two loxodromic elements in $SU(3, 1)$ with distinct fixed points. Then corresponding to the fixed points there are three cross-ratios $\mathbb{X}_k(A, B)$, $k = 1, 2, 3$ that determines the four points uniquely. The collection of all such cross-ratios corresponding to pair of loxodromic elements form a variety, called the *cross-ratio variety*. It follows that every point in this variety has five real degrees of freedom, see Proposition 3.1. For more details on cross ratios in the geometry of rank one symmetric spaces, see Platis [17].

The pair (A, B) is called *non-singular* if

- (i) A and B are loxodromics without a common fixed point.
- (ii) The fixed points of A and B do not lie on a common \mathbb{C}^2 -chain.

- (iii) The fixed point set of A is disjoint from at least one of the \mathbb{C}^2 -chains polar to the positive eigenvectors of B and, the fixed point set of B is disjoint from at least one of the \mathbb{C}^2 -chains polar to the positive eigenvectors of A .

Condition (iii) can also be stated in terms of the Goldman's invariants introduced in Section 4. It is equivalent to the condition that for some $i, j \in \{1, 2\}$, $\eta_i(A, B) \neq 0$ and $\nu_j(A, B) \neq 0$, see Section 5.

The free subgroup $\langle A, B \rangle$ is non-singular if the generating pair is non-singular. In particular, a non-singular subgroup is Zariski-dense in $\mathrm{SU}(3, 1)$. To a non-singular pair (A, B) , we associate a pair of complex numbers $\alpha_i(A, B)$ and $\beta_j(A, B)$ which are given by the following:

$$\alpha_1(A, B) = \mathbb{X}(\mathbf{r}_A, \mathbf{a}_A, \mathbf{x}_B, \mathbf{a}_B), \quad \alpha_2(A, B) = \mathbb{X}(\mathbf{r}_A, \mathbf{a}_A, \mathbf{y}_B, \mathbf{a}_B).$$

$$\beta_1(A, B) = \mathbb{X}(\mathbf{r}_B, \mathbf{a}_B, \mathbf{x}_A, \mathbf{a}_A), \quad \beta_2(A, B) = \mathbb{X}(\mathbf{r}_B, \mathbf{a}_B, \mathbf{y}_A, \mathbf{a}_A),$$

where $\mathbb{X}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4)$ is given by (1.1). We shall refer to $\alpha_1(A, B)$ or $\alpha_2(A, B)$ by α -invariant and, $\beta_1(A, B)$ or $\beta_2(A, B)$ by β -invariant. Condition (iii) in the above definition ensures that there exist at least one non-zero α -invariant and one non-zero β -invariant for non-singular subgroups. We prove the following:

Theorem 1.1. *Let A, B be two loxodromic elements in $\mathrm{SU}(3, 1)$ such that they generate a non-singular subgroup $\langle A, B \rangle$. Then $\langle A, B \rangle$ is determined up to conjugacy by the following parameters:*

$\mathrm{tr}(A)$, $\mathrm{tr}(B)$, $\sigma(A)$, $\sigma(B)$, $\mathbb{X}_k(A, B)$, $k = 1, 2, 3$, any one of the non-zero α -invariants and any one of the non-zero β -invariants, where $\mathrm{tr}(A) = \mathrm{trace}(A)$, $\sigma(A) = \frac{1}{2}(\mathrm{tr}^2(A) - \mathrm{tr}(A^2))$.

In the parameter space associated to $\langle A, B \rangle$, the parameters $\mathrm{tr}(A)$, $\mathrm{tr}(B)$, α and β are complex numbers, $\sigma(A)$, $\sigma(B)$ are real numbers, $(\mathbb{X}_1, \mathbb{X}_2, \mathbb{X}_3)$ live on the cross-ratio variety, which is 5 dimensional. Thus we need a total of $(4 \times 2 + 2 \times 1 + 5) = 15$ dimensional real parameters to specify $\langle A, B \rangle$ up to conjugacy. We call these parameters the *Fenchel-Nielsen coordinates* of $\langle A, B \rangle$.

Suppose F_2 is a free group of rank two. Let $F_2 = \langle m, n \rangle$. Let us consider the $\mathrm{SU}(3, 1)$ -deformation space of F_2 : $\mathcal{M} = \mathrm{Hom}(F_2, \mathrm{SU}(3, 1)) // \mathrm{SU}(3, 1)$. Let $\mathcal{R}^{\mathrm{lox}}$ be the subset of $\mathrm{Hom}(F_2, \mathrm{SU}(3, 1))$ defined by

$$\mathcal{R}^{\mathrm{lox}} = \{\rho : F_2 \rightarrow \mathrm{SU}(3, 1) \mid \rho(m) \text{ and } \rho(n) \text{ are loxodromic}\}.$$

For $i, j \in \{1, 2\}$, let

$$\mathcal{R}_{ij}^{\mathrm{lox}} = \{\rho \in \mathcal{R}^{\mathrm{lox}} \mid (\rho(m), \rho(n)) \text{ is non-singular and } \alpha_i(\rho(m), \rho(n)) \neq 0 \neq \beta_j(\rho(m), \rho(n))\}.$$

Let

$$\mathcal{R}_o^{\mathrm{lox}} = \{\rho \in \mathcal{R}^{\mathrm{lox}} \mid (\rho(m), \rho(n)) \text{ is non-singular}\}, \text{ thus}$$

$$\mathcal{R}_o^{\mathrm{lox}} = \mathcal{R}_{11}^{\mathrm{lox}} \cup \mathcal{R}_{12}^{\mathrm{lox}} \cup \mathcal{R}_{21}^{\mathrm{lox}} \cup \mathcal{R}_{22}^{\mathrm{lox}}.$$

Let $\mathcal{M}_{ij}^{\mathrm{lox}} = \mathcal{R}_{ij}^{\mathrm{lox}} // \mathrm{SU}(3, 1)$. Then Theorem 1.1 classifies the representations of $\mathcal{M}_{ij}^{\mathrm{lox}}$.

Corollary 1.2. *Let $\rho : F_2 \rightarrow \mathrm{SU}(3, 1)$ be a representation such that $\rho(m)$, $\rho(n)$ are loxodromic and generates a non-singular subgroup of $\mathrm{SU}(3, 1)$. Then for some $i, j \in \{1, 2\}$, there exists two non-zero complex parameters α_i and β_j such that these, along with coefficients of the characteristic polynomials of $\rho(m)$, $\rho(n)$ and a point on the cross-ratio variety, completely determine ρ up to conjugacy. The real dimension of the parameter space associated to $\mathcal{M}_{ij}^{\mathrm{lox}}$ is 15.*

Let $\mathcal{M}_o^{\mathrm{lox}} = \mathcal{R}_o^{\mathrm{lox}} // \mathrm{SU}(3, 1)$. It follows that $\mathcal{M}_o^{\mathrm{lox}}$ has been covered by four coordinate patches $\mathcal{M}_{ij}^{\mathrm{lox}}$ which are parametrized by a subset in \mathbb{R}^{15} . On the intersection of any two of these coordinate patches, we have a transition map that is induced by a linear transformation of $\mathbb{C}^{3,1}$. For example, on $\mathcal{M}_{11}^{\mathrm{lox}} \cap \mathcal{M}_{12}^{\mathrm{lox}}$, this transition map is induced by the complex linear transformation that fixes $\mathbf{a}_A, \mathbf{r}_A, \mathbf{a}_B$ and rotates \mathbf{x}_B to \mathbf{y}_B .

Let $\mathcal{C} = \{\gamma_j\}$, $j = 1, 2, \dots, 3g - 3$, be a maximal family of simple closed curves on Σ_g such that they are neither homotopically equivalent to each other nor homotopically trivial. We may assume that in \mathcal{C} each curve is actually a geodesic in its homotopy class and hence the homotopy type of

the curves can be considered as an element in $\pi_1(\Sigma_g)$. We also assume that our curve system is *simple*, i.e. there are g of the curves γ_j that correspond to two boundary components of the same three-holed sphere. We consider discrete, faithful representations $\rho : \pi_1(\Sigma_g) \rightarrow \mathrm{SU}(3, 1)$ such that the $3g - 3$ group elements $\rho(\gamma_j)$ are loxodromic. Then each pair of pants in the complement of \mathcal{C} gives rise to a $(0, 3)$ subgroup of $\mathrm{SU}(3, 1)$, i.e a subgroup generated by two loxodromic elements A, B such that AB is also loxodromic. The three boundary curves in a pair of pants is represented by A, B and $B^{-1}A^{-1}$ respectively. For this reason, A, B and $B^{-1}A^{-1}$ are called *peripheral elements* of the two-generator subgroup $\langle A, B \rangle$.

A discrete, faithful, totally loxodromic representation $\rho : \pi_1(\Sigma_g) \rightarrow \mathrm{SU}(3, 1)$ is called *tame* if the resulting $(0, 3)$ groups from the given pant decomposition are all non-singular in $\mathrm{SU}(3, 1)$. We aim to describe $30g - 30$ parameters that specify tame representations up to conjugacy.

Theorem 1.3. *Let Σ_g be a closed surface of genus g with a simple curve system $\mathcal{C} = \{\gamma_j\}$, $j = 1, 2, \dots, 3g - 3$. Let $\rho : \pi_1(\Sigma_g) \rightarrow \mathrm{SU}(3, 1)$ be a tame representation of the surface group $\pi_1(\Sigma_g)$ into $\mathrm{SU}(3, 1)$. Then we need $30g - 30$ real parameters to specify ρ in the deformation space $\mathrm{Hom}(\pi_1(\Sigma_g), \mathrm{SU}(3, 1)) // \mathrm{SU}(3, 1)$.*

Specifically, these coordinates are $4g - 4$ complex traces, $4g - 4$ of the σ -invariants, $2g - 2$ points on the 5 (real) dimensional cross ratio variety, $2g - 2$ of the α invariants, $2g - 2$ of the β -invariants and $3g - 3$ twist-bend parameters subject to $3g - 3$ constraints corresponding to the compatibility conditions.

The idea to prove the above theorem is similar to the construction of Fenchel-Nielsen coordinates of the classical Teichmüller space. The proof follows from Theorem 1.1 combining it with Proposition 6.3 and Proposition 6.5 that determine $(0, 4)$ and $(1, 1)$ groups respectively.

This paper is organized as follows. In Section 2 we recall the notion of complex hyperbolic space and summarize preliminary results about loxodromic isometries that would be needed later on. In Section 3, we study the Koranyi-Riemann cross ratios of a quadruple of distinct points on the ideal boundary of $\mathbf{H}_{\mathbb{C}}^3$. In Section 4 we derive a sufficient condition for the subgroup $\langle A, B \rangle$ to be reducible in terms of the numerical invariants. We determine the non-singular subgroups $\langle A, B \rangle$ in Section 5. We prove Theorem 1.1 in this section. In Section 6, we introduce the twist-bend parameters and show that the $(0, 4)$ group or the $(1, 1)$ groups are determined uniquely up to conjugation by several invariants introduced in Theorem 1.1 along with the twist-bend parameters. In particular, we prove Proposition 6.3 and Proposition 6.5 in this section. We prove Theorem 1.3 in Section 7.

2. PRELIMINARIES

2.1. Complex Hyperbolic Space. Let $V = \mathbb{C}^{3,1}$ be the complex vector space \mathbb{C}^4 equipped with the Hermitian form of signature $(3, 1)$ given by

$$\langle \mathbf{z}, \mathbf{w} \rangle = \mathbf{w}^* H \mathbf{z} = z_1 \bar{w}_4 + z_2 \bar{w}_2 + z_3 \bar{w}_3 + z_4 \bar{w}_1,$$

where $*$ denotes conjugate transpose. The matrix of the Hermitian form is given by

$$H = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

If H' is any other 4×4 Hermitian matrix with signature $(3, 1)$, then there is a 4×4 matrix C so that $C^* H' C = H$.

We consider the following subspaces of $\mathbb{C}^{3,1}$:

$$V_- = \{\mathbf{z} \in \mathbb{C}^{3,1} : \langle \mathbf{z}, \mathbf{z} \rangle < 0\}, \quad V_+ = \{\mathbf{z} \in \mathbb{C}^{3,1} : \langle \mathbf{z}, \mathbf{z} \rangle > 0\},$$

$$V_0 = \{\mathbf{z} - \{\mathbf{0}\} \in \mathbb{C}^{3,1} : \langle \mathbf{z}, \mathbf{z} \rangle = 0\}.$$

A vector \mathbf{z} in $\mathbb{C}^{3,1}$ is called *positive*, *negative* or *null* depending on whether \mathbf{z} belongs to V_+ , V_- or V_0 . Let $\mathbb{P} : \mathbb{C}^{3,1} - \{\mathbf{0}\} \rightarrow \mathbb{CP}^3$ be the canonical projection onto complex projective space.

The complex hyperbolic space $\mathbf{H}_{\mathbb{C}}^3$ is defined to be $\mathbb{P}V_-$. The ideal boundary $\partial\mathbf{H}_{\mathbb{C}}^3$ is $\mathbb{P}V_0$. The canonical projection of a vector \mathbf{z} is given by $z = \mathbb{P}(\mathbf{z}) = (z_1/z_4, z_2/z_4, z_3/z_4)$. Therefore we can write $\mathbf{H}_{\mathbb{C}}^3 = \mathbb{P}(V_-)$ as

$$\mathbf{H}_{\mathbb{C}}^3 = \{(w_1, w_2, w_3) \in \mathbb{C}^3 : 2\Re(w_1) + |w_2|^2 + |w_3|^2 < 0\}.$$

This gives the Siegel domain model of $\mathbf{H}_{\mathbb{C}}^3$. There are two distinguished points in V_0 which we denote by o and ∞ given by

$$o = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \infty = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Then we can write $\partial\mathbf{H}_{\mathbb{C}}^3 = \mathbb{P}(V_0)$ as

$$\partial\mathbf{H}_{\mathbb{C}}^3 - \infty = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : 2\Re(z_1) + |z_2|^2 + |z_3|^2 = 0\}.$$

Conversely, given a point z of $\mathbf{H}_{\mathbb{C}}^3 = \mathbb{P}(V_-) \subset \mathbb{C}P^3$ we may lift $z = (z_1, z_2, z_3)$ to a point \mathbf{z} in V_- , called the standard lift of z , by writing in non-homogeneous coordinates as

$$\mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ 1 \end{bmatrix}.$$

The Bergman metric in $\mathbf{H}_{\mathbb{C}}^3$ is defined in terms of the Hermitian form given by:

$$ds^2 = -\frac{4}{\langle z, z \rangle^2} \det \begin{bmatrix} \langle z, z \rangle & \langle dz, z \rangle \\ \langle z, dz \rangle & \langle dz, dz \rangle \end{bmatrix}.$$

If z and w in $\mathbf{H}_{\mathbb{C}}^3$ correspond to vectors \mathbf{z} and \mathbf{w} in V_- then the Bergman metric is given by the distance ρ :

$$\cosh^2 \left(\frac{\rho(z, w)}{2} \right) = \frac{\langle \mathbf{z}, \mathbf{w} \rangle \langle \mathbf{w}, \mathbf{z} \rangle}{\langle \mathbf{z}, \mathbf{z} \rangle \langle \mathbf{w}, \mathbf{w} \rangle}$$

2.2. Isometries. Let $U(3, 1)$ be the group of matrices which preserve the Hermitian form $\langle \cdot, \cdot \rangle$. Each such matrix A satisfies the relation $A^{-1} = H^{-1} A^* H$ where A^* is the conjugate transpose of A . The holomorphic isometry group of $\mathbf{H}_{\mathbb{C}}^3$ is the projective unitary group $\text{PSU}(3, 1) = \text{SU}(3, 1) / \{\pm I, \pm iI\}$. It is often more convenient to lift to the four-fold covering $\text{SU}(3, 1)$ to look at the action of the isometries.

Based on their fixed points, holomorphic isometries of $\mathbf{H}_{\mathbb{C}}^3$ are classified as follows:

- (1) An isometry is *elliptic* if it fixes at least one point of $\mathbf{H}_{\mathbb{C}}^3$.
- (2) An isometry is *loxodromic* if it is non-elliptic and fixes exactly two points of $\partial\mathbf{H}_{\mathbb{C}}^3$, one of which is attracting and other repelling.
- (3) An isometry is *parabolic* if it is non-elliptic and fixes exactly one point of $\partial\mathbf{H}_{\mathbb{C}}^3$.

For more details on isometries of $\mathbf{H}_{\mathbb{C}}^3$, see [10].

2.3. Loxodromic Isometries. Let $A \in \text{SU}(3, 1)$ represents a loxodromic isometry. Then A has eigenvalues of the form $re^{i\theta}$, $r^{-1}e^{i\theta}$, $e^{i\phi}$, $e^{-i(2\theta+\phi)}$. We can assume $\theta, \phi \in (-\pi, \pi]$ and $\theta \leq \phi$. Then $(r, \theta, \phi) \in S$, where S is the region defined by:

$$S = \{(r, \theta, \phi) \in \mathbb{R}^3 : r > 1, \theta, \phi \in (-\pi, \pi], \theta \geq \phi\}.$$

Let $a_A \in \partial\mathbf{H}_{\mathbb{C}}^3$ be the attractive fixed point of A . then any lift \mathbf{a}_A of a_A to V_0 is an eigenvector of A and corresponding eigenvalue is $re^{i\theta}$. Similarly if $r_A \in \partial\mathbf{H}_{\mathbb{C}}^3$ is the repelling fixed point of A , then any lift \mathbf{r}_A of r_A to V_0 is an eigenvector of A with eigenvalue $r^{-1}e^{i\theta}$.

For $(r, \theta, \phi) \in S$, define $E(r, \theta, \phi)$ as

$$(2.1) \quad E(r, \theta, \phi) = \begin{bmatrix} re^{i\theta} & & & \\ & e^{i\phi} & & \\ & & e^{-i(2\theta+\phi)} & \\ & & & r^{-1}e^{i\theta} \end{bmatrix}.$$

It is easy to see that $E = E(r, \theta, \phi) \in \text{SU}(3, 1)$ represent a loxodromic map with attractive fixed point $a_E = \infty$ and repelling fixed point $r_E = o$. Equivalently, (2.1) can be represented in the form

$$E(\lambda, \psi) = \text{diag}(e^\lambda, e^{-i\psi + \frac{\bar{\lambda} - \lambda}{2}}, e^{i\psi + \frac{\bar{\lambda} - \lambda}{2}}, e^{-\bar{\lambda}}), \quad \lambda = l + i\theta.$$

Let $\mathbf{x}_A, \mathbf{y}_A$ be the eigenvectors corresponding to the eigenvalues $e^{i\phi}, e^{-i(2\theta+\phi)}$ respectively, scaled so that $\langle \mathbf{x}_A, \mathbf{x}_A \rangle = 1 = \langle \mathbf{y}_A, \mathbf{y}_A \rangle$. Let $C_A = \begin{bmatrix} \mathbf{a}_A & \mathbf{x}_A & \mathbf{y}_A & \mathbf{r}_A \end{bmatrix}$ be the 4×4 matrix, where the lifts \mathbf{a}_A and \mathbf{r}_A are chosen so that C_A has determinant 1. Then $C_A \in \text{SU}(3, 1)$ and $A = C_A E_A(r, \theta, \phi) C_A^{-1}$, where $E_A(r, \theta, \phi)$ is given by (2.1).

Lemma 2.1. *Let $A \in \text{SU}(3, 1)$. Then A has characteristic polynomial*

$$\chi_A(X) = X^4 - \tau_A X^3 + \sigma_A X^2 - \bar{\tau}_A X + 1,$$

where $\tau_A = \text{tr}(A)$ and $\sigma_A = \frac{1}{2}(\text{tr}^2(A) - \text{tr}(A^2))$. Moreover σ_A is real.

For a proof see [9]. We also denote σ_A by $\sigma(A)$ in the sequel.

Proposition 2.2. *Two loxodromic elements in $\text{SU}(3, 1)$ are conjugate if and only if they have the same eigenvalues.*

For a proof see [1]. An immediate consequence of Lemma 2.1 and Proposition 2.2 is:

Corollary 2.3. *Two loxodromic elements A and A' in $\text{SU}(3, 1)$ are conjugate if and only if $\tau_A = \tau_{A'}$ and $\sigma_A = \sigma_{A'}$.*

Lemma 2.4. *Let $A = [Ae_1, Ae_2, Ae_3, Ae_4] \in \text{SU}(3, 1)$, then the vector Ae_3 is uniquely determined by the vectors Ae_1, Ae_2 and Ae_4 .*

Proof. Let W be the subspace spanned by Ae_1, Ae_2, Ae_4 . Let W^\perp be the orthogonal complement of W in $\mathbb{C}^{3,1}$. Observe that since $A \in \text{SU}(3, 1)$, $W \cap W^\perp = \{0\}$ and $W^\perp \neq \{0\}$ is an one dimensional subspace of \mathbb{C}^4 . Let $W^\perp = \langle w \rangle$ for some $w \in \mathbb{C}^4$. Then $Ae_3 \in W^\perp$ implies that $Ae_3 = \lambda w$ for some $\lambda \in \mathbb{C}$. Further the condition $\det(A) = 1$ determines λ uniquely and the assertion follows. \square

Corollary 2.5. *Let $A = [Ae_1, Ae_2, Ae_3, Ae_4]$, $B = [Be_1, Be_2, Be_3, Be_4] \in \text{SU}(3, 1)$ and $C \in \text{SU}(3, 1)$ be such that $CAe_i = Be_i$ for $i = 1, 2, 4$, then $CAe_3 = Be_3$.*

From Lemma 2.4 and Corollary 2.3 we have the following.

Corollary 2.6. *Let A and A' are two loxodromic elements in $\text{SU}(3, 1)$ such that $\tau_A = \tau_{A'}$, $\sigma_A = \sigma_{A'}$, $a_A = a_{A'}$, $r_A = r_{A'}$ and $x_A = x_{A'}$, then $A = A'$.*

3. THE CROSS-RATIOS

3.1. The Koranyi-Riemann cross-ratio. Given a quadruple of distinct points (z_1, z_2, z_3, z_4) on $\partial\mathbf{H}_{\mathbb{C}}^3$, their Koranyi-Riemann cross ratio is defined by

$$\mathbb{X}(z_1, z_2, z_3, z_4) = [z_1, z_2, z_3, z_4] = \frac{\langle \mathbf{z}_3, \mathbf{z}_1 \rangle \langle \mathbf{z}_4, \mathbf{z}_2 \rangle}{\langle \mathbf{z}_4, \mathbf{z}_1 \rangle \langle \mathbf{z}_3, \mathbf{z}_2 \rangle},$$

where, for $i = 1, 2, 3, 4$, \mathbf{z}_i , are lifts of z_i . It can be seen easily that \mathbb{X} is independent of the chosen lifts of z_i 's. By choosing different ordering of the four points, we may define other cross ratios and it can be seen in [5, p.225] that there are certain symmetries that are associated with certain

permutations. After taking these into account, there are only three cross-ratios that remain. Given distinct points z_1, z_2, z_3, z_4 in $\partial\mathbf{H}_{\mathbb{C}}^3$, we define :

$$(3.1) \quad \mathbb{X}_1 = [z_1, z_2, z_3, z_4], \mathbb{X}_2 = [z_1, z_3, z_2, z_4], \mathbb{X}_3 = [z_2, z_3, z_1, z_4]$$

Parker and Platis [15], also see Falbel [4], have shown that the triples of cross-ratios of an ordered quadruple of points in $\partial\mathbf{H}_{\mathbb{C}}^3$ satisfy two real equations. If the ordered triples of points belongs to $\partial\mathbf{H}_{\mathbb{C}}^3$, the corresponding cross-ratios satisfy only one real equation and one real inequality as shown in the following proposition.

Proposition 3.1. *Let z_1, z_2, z_3, z_4 be four distinct points in $\partial\mathbf{H}_{\mathbb{C}}^3$. Let $\mathbb{X}_1, \mathbb{X}_2, \mathbb{X}_3$ be defined by 3.1, then*

$$(3.2) \quad |\mathbb{X}_2| = |\mathbb{X}_1||\mathbb{X}_3|.$$

$$(3.3) \quad 2|\mathbb{X}_1|^2\Re(\mathbb{X}_3) \geq |\mathbb{X}_1|^2 + |\mathbb{X}_2|^2 + 1 - 2\Re(\mathbb{X}_1 + \mathbb{X}_2).$$

Further, equality holds in (3.3) if and only if either of the following holds.

- (i) z_1, z_2, z_4 lie on the same complex line.
- (ii) z_1, z_3, z_4 lie on the same complex line.
- (iii) z_1, z_2, z_3, z_4 lie on the same complex line.

Proof. Since $\mathrm{SU}(3,1)$ acts doubly transitively on $\partial\mathbf{H}_{\mathbb{C}}^3$, we may suppose $z_2 = \infty$ and $z_3 = \mathbf{o}$. Let $\mathbf{z}_1, \mathbf{z}_4$ be lifts of z_1 and z_4 chosen so that $\langle \mathbf{z}_1, \mathbf{z}_4 \rangle = 1$. We write them in coordinates as

$$\mathbf{z}_1 = \begin{bmatrix} \xi_1 \\ \eta_1 \\ \alpha_1 \\ \delta_1 \end{bmatrix}, \mathbf{z}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \mathbf{z}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \mathbf{z}_4 = \begin{bmatrix} \xi_4 \\ \nu_2 \\ \zeta_4 \\ \delta_4 \end{bmatrix}.$$

Then we have

$$(3.4) \quad 0 = \langle \mathbf{z}_1, \mathbf{z}_1 \rangle = \xi_1 \bar{\delta}_1 + \bar{\xi}_1 \delta_1 + |\eta_1|^2 + |\alpha_1|^2$$

$$(3.5) \quad 1 = \langle \mathbf{z}_4, \mathbf{z}_1 \rangle = \xi_4 \bar{\delta}_1 + \bar{\xi}_4 \delta_1 + \nu_2 \bar{\eta}_1 + \zeta_4 \bar{\alpha}_1$$

$$(3.6) \quad 0 = \langle \mathbf{z}_4, \mathbf{z}_4 \rangle = \xi_4 \bar{\delta}_4 + \bar{\xi}_4 \delta_4 + |\nu_2|^2 + |\zeta_4|^2$$

From the definitions of the cross-ratios we have:

$$\mathbb{X}_1 = [\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3, \mathbf{z}_4] = \frac{\langle \mathbf{z}_3, \mathbf{z}_1 \rangle \langle \mathbf{z}_4, \mathbf{z}_2 \rangle}{\langle \mathbf{z}_4, \mathbf{z}_1 \rangle \langle \mathbf{z}_3, \mathbf{z}_2 \rangle} = \bar{\xi}_1 \delta_4$$

$$\mathbb{X}_2 = [\mathbf{z}_1, \mathbf{z}_3, \mathbf{z}_2, \mathbf{z}_4] = \frac{\langle \mathbf{z}_2, \mathbf{z}_1 \rangle \langle \mathbf{z}_4, \mathbf{z}_3 \rangle}{\langle \mathbf{z}_4, \mathbf{z}_1 \rangle \langle \mathbf{z}_2, \mathbf{z}_3 \rangle} = \xi_4 \bar{\delta}_1$$

$$\mathbb{X}_3 = [\mathbf{z}_2, \mathbf{z}_3, \mathbf{z}_1, \mathbf{z}_4] = \frac{\langle \mathbf{z}_1, \mathbf{z}_2 \rangle \langle \mathbf{z}_4, \mathbf{z}_3 \rangle}{\langle \mathbf{z}_4, \mathbf{z}_2 \rangle \langle \mathbf{z}_1, \mathbf{z}_3 \rangle} = \frac{\xi_4 \delta_1}{\xi_1 \delta_4}$$

We immediately see that

$$|\mathbb{X}_3| = \frac{|\mathbb{X}_2|}{|\mathbb{X}_1|}.$$

Using eqs. 3.4 – 3.6, we have:

$$\begin{aligned}
|\mathbb{X}_1|^2 |\mathbb{X}_3 - 1|^2 &= |\xi_4 \delta_1 - \xi_1 \delta_4|^2 \\
&= |\xi_4 \delta_1|^2 + |\xi_1 \delta_4|^2 - \xi_4 \delta_1 \bar{\xi}_1 \bar{\delta}_4 - \bar{\xi}_4 \bar{\delta}_1 \xi_1 \delta_4 \\
&= |\bar{\xi}_1 \delta_4|^2 + |\xi_4 \bar{\delta}_1|^2 + \xi_4 \bar{\delta}_4 (\xi_1 \bar{\delta}_1 + |\eta_1|^2 + |\alpha_1|^2) + \bar{\xi}_4 \delta_4 (\bar{\xi}_1 \delta_1 + |\eta_1|^2 + |\alpha_1|^2) \\
&= |\bar{\xi}_1 \delta_4 + \xi_4 \bar{\delta}_1|^2 - (|\nu_2|^2 + |\zeta_4|^2)(|\eta_1|^2 + |\alpha_1|^2) \\
&= |\bar{\xi}_1 \delta_4 + \xi_4 \bar{\delta}_1|^2 - |\nu_2 \bar{\alpha}_1 + \zeta_4 \bar{\alpha}_1|^2 - |\nu_2 \alpha_1 - \eta_1 \zeta_4|^2 \\
&= |\mathbb{X}_1 + \mathbb{X}_2|^2 + |1 - \mathbb{X}_1 - \mathbb{X}_2|^2 - |\nu_2 \alpha_1 - \eta_1 \zeta_4|^2
\end{aligned}$$

This implies

$$|\mathbb{X}_1|^2 |\mathbb{X}_3 - 1|^2 - |\mathbb{X}_1 + \mathbb{X}_2|^2 + |1 - \mathbb{X}_1 - \mathbb{X}_2|^2 = -|\nu_2 \alpha_1 - \eta_1 \zeta_4|^2 \leq 0$$

Rearranging this gives the inequality we want. Further the above inequality is an equality if and only if

$$\nu_2 \alpha_1 - \eta_1 \zeta_4 = 0, \text{ i.e. } \frac{\nu_2}{\eta_1} = \frac{\zeta_4}{\alpha_1}.$$

This means either of the conditions (i), (ii), (iii) given in the statement. This proves the proposition. \square

Platis [17] has proved a generalization of the above proposition for arbitrary rank 1 symmetric spaces of non-compact type and has applied it to derive Ptolemaean inequality on the boundary of a rank 1 symmetric space of non-compact type. Since we have restricted ourselves only to three dimensional complex hyperbolic geometry, our proof above is much simpler.

Corollary 3.2. *Let $\mathbb{X}_1, \mathbb{X}_2, \mathbb{X}_3$ be defined by 3.1, then $2\Re(\mathbb{X}_1 + \mathbb{X}_2) \geq 1$.*

Proof. Since $\Re(\mathbb{X}_3) \leq |\mathbb{X}_3|$,

$$\begin{aligned}
2\Re(\mathbb{X}_1 + \mathbb{X}_2) - 1 &\geq |\mathbb{X}_1|^2 + |\mathbb{X}_2|^2 - 2|\mathbb{X}_1|^2 \Re(\mathbb{X}_3) \\
&\geq |\mathbb{X}_1|^2 + |\mathbb{X}_2|^2 - 2|\mathbb{X}_1|^2 |\mathbb{X}_3| \\
&= |\mathbb{X}_1|^2 + |\mathbb{X}_2|^2 - 2|\mathbb{X}_1| |\mathbb{X}_2| \\
&= (|\mathbb{X}_1| - |\mathbb{X}_2|)^2 \geq 0
\end{aligned}$$

In particular $2\Re(\mathbb{X}_1 + \mathbb{X}_2) \geq 1$ \square

3.2. Cartan's angular invariant. Let z_1, z_2, z_3 be three distinct points of $\partial\mathbf{H}_{\mathbb{C}}^3$ with lifts $\mathbf{z}_1, \mathbf{z}_2$ and \mathbf{z}_3 respectively. Cartan's angular invariant is defined as follows :

$$\mathbb{A}(z_1, z_2, z_3) = \arg(-\langle \mathbf{z}_1, \mathbf{z}_2 \rangle \langle \mathbf{z}_2, \mathbf{z}_3 \rangle \langle \mathbf{z}_3, \mathbf{z}_1 \rangle).$$

The angular invariant is invariant under $\text{SU}(3, 1)$ and independent of the chosen lifts. The following proposition shows that this invariant determines any triples of distinct points in $\partial\mathbf{H}_{\mathbb{C}}^3$ up to $\text{SU}(3, 1)$ -equivalence.

Proposition 3.3. *Let z_1, z_2, z_3 and z'_1, z'_2, z'_3 be triples of distinct points of $\partial\mathbf{H}_{\mathbb{C}}^3$. Then $\mathbb{A}(z_1, z_2, z_3) = \mathbb{A}(z'_1, z'_2, z'_3)$ if and only if there exist $A \in \text{SU}(3, 1)$ so that $A(z_j) = z'_j$ for $j = 1, 2, 3$.*

For a proof see [5]. Also we have the following result from [5].

Proposition 3.4. *Let z_1, z_2, z_3 be three distinct points of $\partial\mathbf{H}_{\mathbb{C}}^3$ and let $\mathbb{A} = \mathbb{A}(z_1, z_2, z_3)$ be their angular invariant. Then*

$$(1) \quad \mathbb{A} \in [-\frac{\pi}{2}, \frac{\pi}{2}].$$

- (2) $\mathbb{A} = \pm \frac{\pi}{2}$ if and only if z_1, z_2, z_3 lie on the same chain.
 (3) $\mathbb{A} = 0$ if and only if z_1, z_2, z_3 lie on a totally real totally geodesic subspace.

Proposition 3.5. *Let z_1, \dots, z_4 be distinct points of $\partial\mathbf{H}_{\mathbb{C}}^3$ and let $\mathbb{X}_1, \mathbb{X}_2, \mathbb{X}_3$ denote the cross-ratios defined by 3.1. Suppose $\mathbb{X}_1, \mathbb{X}_2$ and \mathbb{X}_3 are non-real complex numbers. Let $\mathbb{A}_1 = \mathbb{A}(z_4, z_3, z_2)$ and $\mathbb{A}_2 = \mathbb{A}(z_3, z_2, z_1)$. Then*

- (1) $\mathbb{A}_1 + \mathbb{A}_2 = \arg(\overline{\mathbb{X}_1}\mathbb{X}_2)$.
 (2) $\mathbb{A}_1 - \mathbb{A}_2 = \arg(\mathbb{X}_3)$.

Note that the above proposition is not true if \mathbb{X}_i 's are real numbers. Cuhna-Gusevskii [2, p.279] have given a counter-example to the above proposition when \mathbb{X}_i 's are real numbers. However, when all the cross-ratios are non-real complex numbers, the argument as in the proof of [15, Proposition 5.8] goes through and we have the above proposition. An explanation that the proof of [15, Proposition 5.8] does not carry over to the real cross ratio case is that the principal argument of complex numbers is a well-defined function from $\mathbb{C} - \{0\}$ to the semi-open interval $(-\pi, \pi]$. On the other hand, $\mathbb{A}_1 \pm \mathbb{A}_2$ are well-defined functions from distinct triple points on $\partial\mathbf{H}_{\mathbb{C}}^3$ onto the closed interval $[-\pi, \pi]$. So, the principal argument can not be identified with $\mathbb{A}_1 \pm \mathbb{A}_2$, especially on the boundary points of the intervals and those cases correspond when the cross ratios are real numbers.

Proposition 3.6. *Let z_1, z_2, z_3, z_4 be distinct points of $\partial\mathbf{H}_{\mathbb{C}}^3$ with complex non-real cross ratios $\mathbb{X}_1, \mathbb{X}_2, \mathbb{X}_3$. Let z'_1, z'_2, z'_3, z'_4 be another set of distinct points of $\partial\mathbf{H}_{\mathbb{C}}^3$ with corresponding cross ratios $\mathbb{X}'_1, \mathbb{X}'_2, \mathbb{X}'_3$. If $\mathbb{X}'_i = \mathbb{X}_i$ for $i = 1, 2, 3$, then there exist $A \in \text{SU}(3, 1)$ such that $A(z_j) = z'_j$ for $j = 1, 2, 3, 4$.*

Proof. Since $\text{SU}(3, 1)$ acts doubly transitively on $\partial\mathbf{H}_{\mathbb{C}}^3$, we may assume without loss of generality that $z_2 = z'_2 = \infty$, $z_3 = z'_3 = o$. We write the lifts of other points as

$$\mathbf{z}_1 = \begin{bmatrix} \xi_1 \\ \eta_1 \\ \alpha_1 \\ \delta_1 \end{bmatrix}, \mathbf{z}_4 = \begin{bmatrix} \xi_4 \\ \nu_2 \\ \zeta_4 \\ \delta_4 \end{bmatrix}, \mathbf{z}'_1 = \begin{bmatrix} \xi'_1 \\ \eta'_1 \\ \alpha'_1 \\ \delta'_1 \end{bmatrix}, \mathbf{z}'_4 = \begin{bmatrix} \xi'_4 \\ \nu'_2 \\ \zeta'_4 \\ \delta'_4 \end{bmatrix}.$$

We may suppose that lifts of these points are chosen so that $\langle \mathbf{z}_4, \mathbf{z}_1 \rangle = \langle \mathbf{z}'_4, \mathbf{z}'_1 \rangle$, i.e

$$\bar{\xi}_1\delta_4 + \xi_4\bar{\delta}_1 + \nu_2\bar{\eta}_1 + \zeta_4\bar{\alpha}_1 = \bar{\xi}'_1\delta'_4 + \xi'_4\bar{\delta}'_1 + \nu'_2\bar{\eta}'_1 + \zeta'_4\bar{\alpha}'_1.$$

Then our condition on the cross-ratios gives :

$$\begin{aligned} \bar{\xi}_1\delta_4 &= \bar{\xi}'_1\delta'_4, \\ \xi_4\bar{\delta}_1 &= \xi'_4\bar{\delta}'_1, \\ \frac{\xi_4\delta_1}{\xi_1\delta_4} &= \frac{\xi'_4\delta'_1}{\xi'_1\delta'_4}. \end{aligned}$$

Hence we also have

$$(3.7) \quad \nu_2\bar{\eta}_1 + \zeta_4\bar{\alpha}_1 = \nu'_2\bar{\eta}'_1 + \zeta'_4\bar{\alpha}'_1$$

Let us denote the angular invariants of the points by $\mathbb{A}_1 = \mathbb{A}(z_4, z_3, z_2)$, $\mathbb{A}_2 = \mathbb{A}(z_3, z_2, z_1)$, $\mathbb{A}'_1 = \mathbb{A}(z'_4, z'_3, z'_2)$, $\mathbb{A}'_2 = \mathbb{A}(z'_3, z'_2, z'_1)$. Using Proposition 3.5, we see that $\mathbb{A}_1 + \mathbb{A}_2 = \mathbb{A}'_1 + \mathbb{A}'_2$ and $\mathbb{A}_1 - \mathbb{A}_2 = \mathbb{A}'_1 - \mathbb{A}'_2$. Hence $\mathbb{A}_1 = \mathbb{A}'_1$ and $\mathbb{A}_2 = \mathbb{A}'_2$. From Proposition 3.3, we see that there exist $A_1, A_2 \in \text{SU}(3, 1)$ such that $A_1(z_2) = z'_2, A_1(z_3) = z'_3, A_1(z_4) = z'_4$ and $A_2(z_1) = z'_1, A_2(z_2) = z'_2, A_2(z_3) = z'_3$.

Because A_1 fixes $z_2 = \infty$ and $z_3 = 0$, it is of form

$$\begin{bmatrix} \lambda & & \\ & U_1 & \\ & & \bar{\lambda}^{-1} \end{bmatrix}$$

where $|\lambda| \neq 1$ and $U_1 \in U(2)$. Hence we have $\lambda\xi_4 = \xi'_4$, $\bar{\lambda}^{-1}\delta_4 = \delta'_4$ and $U_1 \begin{bmatrix} \nu_2 \\ \zeta_4 \end{bmatrix} = \begin{bmatrix} \nu'_2 \\ \zeta'_4 \end{bmatrix}$. Therefore

$$\begin{aligned} \xi'_1 &= \frac{\bar{\delta}_4}{\delta'_4} \xi_1 \\ &= \lambda \xi_1 \\ \delta'_1 &= \delta_1 \frac{\bar{\xi}_4}{\xi'_4} \\ &= \bar{\lambda}^{-1} \delta_1. \end{aligned}$$

Hence A_2 is of form

$$\begin{bmatrix} \lambda & & \\ & U_2 & \\ & & \bar{\lambda}^{-1} \end{bmatrix}$$

where, $U_2 \in U(2)$ so that

$$U_2 \begin{bmatrix} \eta_1 \\ \alpha_1 \end{bmatrix} = \begin{bmatrix} \eta'_1 \\ \alpha'_1 \end{bmatrix}.$$

It is enough to prove that there exist $U \in U(2)$ such that

$$U \begin{bmatrix} \nu_2 \\ \zeta_4 \end{bmatrix} = \begin{bmatrix} \nu'_2 \\ \zeta'_4 \end{bmatrix} \text{ and } U \begin{bmatrix} \eta_1 \\ \alpha_1 \end{bmatrix} = \begin{bmatrix} \eta'_1 \\ \alpha'_1 \end{bmatrix}.$$

$$\text{Let us denote by } \mathbf{y}_1 = \begin{bmatrix} \eta_1 \\ \alpha_1 \end{bmatrix}, \mathbf{y}_4 = \begin{bmatrix} \nu_2 \\ \zeta_4 \end{bmatrix}, \mathbf{y}_1' = \begin{bmatrix} \eta'_1 \\ \alpha'_1 \end{bmatrix}, \mathbf{y}_4' = \begin{bmatrix} \nu'_2 \\ \zeta'_4 \end{bmatrix}.$$

From (3.7), we have

$$(3.8) \quad \ll \mathbf{y}_4, \mathbf{y}_1 \gg = \ll \mathbf{y}_4', \mathbf{y}_1' \gg$$

where $\ll \cdot, \cdot \gg$ is the standard positive-definite Hermitian form on \mathbb{C}^2 . Also we have $U_1 \mathbf{y}_4 = \mathbf{y}_4'$ and $U_2 \mathbf{y}_1 = \mathbf{y}_1'$. Then $U_1, U_2 \in U(2)$ implies

$$(3.9) \quad \ll \mathbf{y}_4, \mathbf{y}_4 \gg = \ll \mathbf{y}_4', \mathbf{y}_4' \gg$$

$$(3.10) \quad \ll \mathbf{y}_1, \mathbf{y}_1 \gg = \ll \mathbf{y}_1', \mathbf{y}_1' \gg$$

Suppose \mathbf{y}_1 and \mathbf{y}_4 are linearly independent over \mathbb{C} and so forms a basis of \mathbb{C}^2 . Let U be the 2×2 matrix so that $U\mathbf{y}_1 = \mathbf{y}_1'$ and $U\mathbf{y}_4 = \mathbf{y}_4'$. Then from (3.8) – (3.10) it follows that U preserves the Hermitian form $\ll \cdot, \cdot \gg$ on \mathbb{C}^2 , so $U \in U(2)$ and we are done.

Now consider the case when \mathbf{y}_1 and \mathbf{y}_4 are linearly dependent over \mathbb{C} i.e. $\mathbf{y}_4 = \mu \mathbf{y}_1$ for some $\mu \in \mathbb{C}$. Then since the form $\ll \cdot, \cdot \gg$ is positive definite and using 3.8–3.10, this is true if and only if

$$\begin{aligned} &\ll \mathbf{y}_4 - \mu \mathbf{y}_1, \mathbf{y}_4 - \mu \mathbf{y}_1 \gg = \mathbf{0} \\ &\Leftrightarrow \ll \mathbf{y}_4' - \mu \mathbf{y}_1', \mathbf{y}_4' - \mu \mathbf{y}_1' \gg = \mathbf{0} \\ &\Leftrightarrow \mathbf{y}_4' = \mu \mathbf{y}_1' \end{aligned}$$

Therefore either of U_1 and U_2 works and this completes the proof. \square

3.2.1. *When cross ratios are all real.* Suppose all the three cross-ratios are real. Then (3.2) implies $\mathbb{X}_3 = \pm \mathbb{X}_2/\mathbb{X}_1$. The following result can be proved along the same line as in the proof of [15, Proposition 5.12].

Lemma 3.7. *Suppose $\mathbb{X}_1, \mathbb{X}_2$ and \mathbb{X}_3 are all real.*

- (1) *If $\mathbb{X}_3 = -\mathbb{X}_2/\mathbb{X}_1$, then the points z_j all lie on a chain.*
- (2) *If $\mathbb{X}_3 = \mathbb{X}_2/\mathbb{X}_1$, then the points z_j all lie in a totally real Lagrangian subspace.*

The following result follows from [5, p.225].

Lemma 3.8. *Suppose z_1, z_2, z_3 and z_4 all lie on the same chain. Then $\mathbb{X}_1, \mathbb{X}_2$ and \mathbb{X}_3 are each real.*

Lemma 3.9. *If z_1, z_2, z_3, z_4 are contained in the same totally real totally geodesic subspace, then $\mathbb{X}_1, \mathbb{X}_2, \mathbb{X}_3$ are real numbers.*

Proof. Let ι be the anti-holomorphic involution fixing the totally real totally geodesic subspace. Then for $i = 1, 2, 3$, applying ι we get $\mathbb{X}_i = \overline{\mathbb{X}_i}$. Hence all the cross-ratios are real. \square

Summarizing the above lemmas we have the following.

Proposition 3.10. *Let z_1, z_2, z_3, z_4 are distinct points on $\partial\mathbf{H}_{\mathbb{C}}^3$. Then the cross ratios $\mathbb{X}_1, \mathbb{X}_2$ and \mathbb{X}_3 are real numbers if and only if z_1, z_2, z_3, z_4 all lie on the same chain or the same totally real totally geodesic subspace.*

4. A SUFFICIENT CONDITION FOR IRREDUCIBILITY

Let A, B be loxodromic elements in $\mathrm{SU}(3, 1)$ and following the notation of Section 2.3, let

$$C_A = \begin{bmatrix} \mathbf{a}_A & \mathbf{x}_A & \mathbf{y}_A & \mathbf{r}_A \end{bmatrix}, C_B = \begin{bmatrix} \mathbf{a}_B & \mathbf{x}_B & \mathbf{y}_B & \mathbf{r}_B \end{bmatrix}$$

be the eigen matrices associated with A and B respectively. The Koranyi-Riemann cross-ratios of A and B are defined by

$$(4.1) \quad \mathbb{X}_1(A, B) = [a_B, a_A, r_A, r_B] = \frac{\langle \mathbf{r}_A, \mathbf{a}_B \rangle \langle \mathbf{r}_B, \mathbf{a}_A \rangle}{\langle \mathbf{r}_B, \mathbf{a}_B \rangle \langle \mathbf{r}_A, \mathbf{a}_A \rangle}$$

$$(4.2) \quad \mathbb{X}_2(A, B) = [a_B, r_A, a_A, r_B] = \frac{\langle \mathbf{a}_A, \mathbf{a}_B \rangle \langle \mathbf{r}_B, \mathbf{r}_A \rangle}{\langle \mathbf{r}_B, \mathbf{a}_B \rangle \langle \mathbf{a}_A, \mathbf{r}_A \rangle}$$

$$(4.3) \quad \mathbb{X}_3(A, B) = [a_A, r_A, a_B, r_B] = \frac{\langle \mathbf{a}_B, \mathbf{a}_A \rangle \langle \mathbf{r}_B, \mathbf{r}_A \rangle}{\langle \mathbf{r}_B, \mathbf{a}_A \rangle \langle \mathbf{a}_B, \mathbf{r}_A \rangle}$$

In [5], Goldman defines η -invariant for a triple of points with two points on $\partial\mathbf{H}_{\mathbb{C}}^3$ and one point on $\mathbb{P}(V_+)$. Following Goldman's definition, we define η -invariants associated to A and B as follows

$$\eta_1(A, B) = \eta(a_A, r_A; x_B) = \frac{\langle \mathbf{a}_A, \mathbf{x}_B \rangle \langle \mathbf{x}_B, \mathbf{r}_A \rangle}{\langle \mathbf{a}_A, \mathbf{r}_A \rangle \langle \mathbf{x}_B, \mathbf{x}_B \rangle}$$

$$\eta_2(A, B) = \eta(a_A, r_A; y_B) = \frac{\langle \mathbf{a}_A, \mathbf{y}_B \rangle \langle \mathbf{y}_B, \mathbf{r}_A \rangle}{\langle \mathbf{a}_A, \mathbf{r}_A \rangle \langle \mathbf{y}_B, \mathbf{y}_B \rangle}$$

$$\nu_1(A, B) = \eta(a_B, r_B; x_A) = \frac{\langle \mathbf{a}_B, \mathbf{x}_A \rangle \langle \mathbf{x}_A, \mathbf{r}_B \rangle}{\langle \mathbf{a}_B, \mathbf{r}_B \rangle \langle \mathbf{x}_A, \mathbf{x}_A \rangle}$$

$$\nu_2(A, B) = \eta(a_B, r_B; y_A) = \frac{\langle \mathbf{a}_B, \mathbf{y}_A \rangle \langle \mathbf{y}_A, \mathbf{r}_B \rangle}{\langle \mathbf{a}_B, \mathbf{r}_B \rangle \langle \mathbf{y}_A, \mathbf{y}_A \rangle}$$

We define

$$\zeta_o(A, B) = [y_A, x_A, x_B, y_B] = \frac{\langle \mathbf{x}_B, \mathbf{y}_A \rangle \langle \mathbf{y}_B, \mathbf{x}_A \rangle}{\langle \mathbf{x}_B, \mathbf{x}_A \rangle \langle \mathbf{y}_B, \mathbf{y}_A \rangle}$$

It is clear from the definition that the \mathbb{X}_i 's, η_j 's and ζ_o are conjugacy invariants for the two generator subgroup $\langle A, B \rangle$ of $\mathrm{SU}(3, 1)$ and their values are independent of the chosen lifts of eigenvectors.

Theorem 4.1. *Let $\langle A, B \rangle$ be a discrete, free subgroup of $\mathrm{SU}(3, 1)$ that is generated by two loxodromic elements A and B . Then $\langle A, B \rangle$ preserves a \mathbb{C}^2 -plane if and only if one of the following holds.*

- (i) $\zeta_o = 0$ and, either $\eta_1(A, B) = 0 = \nu_1(A, B)$ or $\eta_2(A, B) = 0 = \nu_2(A, B)$.
- (ii) $\zeta_o = \infty$ and, either $\eta_1(A, B) = 0 = \nu_2(A, B)$ or $\eta_2(A, B) = 0 = \nu_1(A, B)$.

Proof. Note that a two dimensional totally geodesic subspace of $\mathbf{H}_{\mathbb{C}}^3$ corresponds to a copy of $\mathbb{C}^{2,1}$.

The condition is necessary. Suppose $\langle A, B \rangle$ preserve a copy of $\mathbb{C}^{2,1}$. Observe that $\langle A, B \rangle$ preserve a copy of $\mathbb{C}^{2,1}$ if and only if A and B have a common space-like eigen-vector. Thus, either of the following cases arises:

- (a) $x_A = x_B$
- (b) $y_A = y_B$
- (c) $y_A = x_B$
- (d) $x_A = y_B$

The result follows from the definition of $\eta_i(A, B)$'s and $\zeta_o(A, B)$.

The condition is sufficient. Suppose $\zeta_o = 0$. We discuss the case (i) i.e. let

$$\eta_1(A, B) = 0 = \nu_1(A, B) = 0 = \zeta_o(A, B).$$

We claim that $x_A = x_B$. We have

$$\begin{aligned} \langle \mathbf{a}_A, \mathbf{x}_B \rangle \langle \mathbf{x}_B, \mathbf{r}_A \rangle &= 0 \\ \langle \mathbf{a}_B, \mathbf{x}_A \rangle \langle \mathbf{x}_A, \mathbf{r}_B \rangle &= 0 \\ \langle \mathbf{x}_B, \mathbf{y}_A \rangle \langle \mathbf{y}_B, \mathbf{x}_A \rangle &= 0 \end{aligned}$$

Different subcases arises, it is enough to consider the following subcase

$$(4.4) \quad \langle \mathbf{a}_A, \mathbf{x}_B \rangle = 0, \langle \mathbf{a}_B, \mathbf{x}_A \rangle = 0, \langle \mathbf{y}_B, \mathbf{x}_A \rangle = 0.$$

Since, $\{\mathbf{a}_B, \mathbf{x}_B, \mathbf{y}_B, \mathbf{r}_B\}$ is a basis for $\mathbb{C}^{3,1}$, hence there exists scalars $\mu_1, \mu_2, \mu_3, \mu_4$ such that

$$\mathbf{x}_A = \mu_1 \mathbf{a}_B + \mu_2 \mathbf{x}_B + \mu_3 \mathbf{y}_B + \mu_4 \mathbf{r}_B.$$

The conditions $\langle \mathbf{a}_B, \mathbf{x}_A \rangle = 0 = \langle \mathbf{y}_B, \mathbf{x}_A \rangle$ implies $\mu_3 = 0 = \mu_4$. Hence

$$\mathbf{x}_A = \mu_1 \mathbf{a}_B + \mu_2 \mathbf{x}_B.$$

This implies

$$0 = \langle \mathbf{x}_A, \mathbf{a}_A \rangle = \mu_1 \langle \mathbf{a}_B, \mathbf{a}_A \rangle + \mu_2 \langle \mathbf{x}_B, \mathbf{a}_A \rangle.$$

Using (4.4) we have $\mu_1 \langle \mathbf{a}_B, \mathbf{a}_A \rangle = 0$. Since $\langle \mathbf{a}_B, \mathbf{a}_A \rangle \neq 0$, we have $\mu_1 = 0$. Hence $\mathbf{x}_A = \mu_2 \mathbf{x}_B$ i.e. $x_B = x_A$, proving the result for the case (i). The argument in the other cases are similar.

Note that if $\zeta_o = \infty$, then $1/\zeta_o = 0$ and similar arguments work in these cases also. \square

The subgroup $\langle A, B \rangle$ of $\mathrm{SU}(3, 1)$ is called *irreducible* or *Zariski-dense* if it does not preserves a totally geodesic subspace of $\mathbf{H}_{\mathbb{C}}^3$. Using the above theorem and the results on cross ratios, it is possible to derive many conditions for irreducibility of $\langle A, B \rangle$. As a special case we have the following.

Corollary 4.2. *Let A and B be two loxodromic elements in $\mathrm{SU}(3, 1)$ such that $\langle A, B \rangle$ is non-singular. Then $\langle A, B \rangle$ is irreducible.*

5. PROOF OF THEOREM 1.1

In this section we follow the notations from Section 2.3. First we shall show that for a non-singular pair (A, B) one can always get a well-defined α -invariant and a well-defined β -invariant.

5.1. α and β -invariants are well-defined. Let A and B be two loxodromics such that they form a non-singular pair. Without loss of generality, we can assume A is a diagonal matrix, that is $C_A = [\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4]$, where $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ is the standard orthonormal basis of $\mathbb{C}^{3,1}$. Let $B = C_B E(\lambda, \psi) C_B^{-1}$, where $C_B = [\mathbf{a}_B, \mathbf{x}_B, \mathbf{y}_B, \mathbf{r}_B]$. Let

$$\mathbf{a}_B = \begin{bmatrix} a \\ e \\ j \\ n \end{bmatrix}, \quad \mathbf{x}_B = \begin{bmatrix} b \\ f \\ k \\ s \end{bmatrix}, \quad \mathbf{y}_B = \begin{bmatrix} c \\ g \\ l \\ p \end{bmatrix}, \quad \mathbf{r}_B = \begin{bmatrix} d \\ h \\ m \\ q \end{bmatrix}.$$

Now we see that

$$\alpha_1(A, B) = \frac{nb}{as}, \quad \alpha_2(A, B) = \frac{nc}{ap},$$

$$\beta_1(A, B) = \frac{\bar{n}\bar{h}}{\bar{q}\bar{e}}, \quad \beta_2(A, B) = \frac{\bar{n}\bar{m}}{\bar{q}\bar{j}}.$$

Since \mathbf{a}_B and \mathbf{r}_B are negative vectors, we must have a, n and q non-zeros. Now note that

$$(5.1) \quad \langle \mathbf{a}_A, \mathbf{x}_B \rangle = b, \quad \langle \mathbf{r}_A, \mathbf{x}_B \rangle = s,$$

$$(5.2) \quad \langle \mathbf{a}_A, \mathbf{y}_B \rangle = c, \quad \langle \mathbf{r}_A, \mathbf{y}_B \rangle = p,$$

$$(5.3) \quad \langle \mathbf{a}_B, \mathbf{x}_A \rangle = e, \quad \langle \mathbf{r}_B, \mathbf{x}_A \rangle = h,$$

$$(5.4) \quad \langle \mathbf{a}_B, \mathbf{y}_A \rangle = j, \quad \langle \mathbf{r}_B, \mathbf{y}_A \rangle = m.$$

It follows from condition (iii) of the definition of non-singularity that neither of \mathbf{a}_A and \mathbf{r}_A belong to at least one of the \mathbb{C}^2 -chains \mathbf{x}_B^\perp and \mathbf{y}_B^\perp and also, neither of \mathbf{a}_B and \mathbf{r}_B belong to one of the \mathbb{C}^2 -chains \mathbf{x}_A^\perp and \mathbf{y}_A^\perp . Thus, at least one of the equations (5.1) and (5.2) must have entirely non-zero solution. Similarly, the solution of one of the equations (5.3) and (5.3) will also be entirely non-zero. Thus at least one α -invariant and one β -invariant are always well-defined complex numbers for a non-singular pair of loxodromics.

It can further be seen from the definition of Goldman's eta invariants that the well-definedness of α -invariant and β -invariant can be stated equivalently by saying that for some $i, j \in \{1, 2\}$, $\eta_i(A, B) \neq 0$ and $\nu_j(A, B) \neq 0$.

5.2. Proof of Theorem 1.1.

Lemma 5.1. *Let A, B, A', B' be loxodromic elements in $\mathrm{SU}(3, 1)$. Let $\langle A, B \rangle$ be a non-singular subgroup in $\mathrm{SU}(3, 1)$ such that for some $i, j \in \{1, 2\}$, $\eta_i(A, B) \neq 0$ and $\nu_j(A, B) \neq 0$. Suppose $\alpha_i(A, B) = \alpha_i(A', B')$, $\beta_j(A, B) = \beta_j(A', B')$ and, for $k = 1, 2, 3$, $\mathbb{X}_k(A, B) = \mathbb{X}_k(A', B')$. Then there exist $C \in \mathrm{SU}(3, 1)$ such that $C(a_A) = a_{A'}, C(x_A) = x_{A'}, C(y_A) = y_{A'}, C(r_A) = r_{A'}$ and $C(a_B) = a_{B'}, C(x_B) = x_{B'}, C(y_B) = y_{B'}, C(r_B) = r_{B'}$.*

Proof. We shall prove the lemma assuming that $(i, j) = (1, 1)$. The rest of the cases are similar.

Since $\mathbb{X}_i(A, B) = \mathbb{X}_i(A', B')$, $i = 1, 2, 3$, by Proposition 3.6 it follows that there exist $C \in \mathrm{SU}(3, 1)$ such that $a_{A'} = C(a_A)$, $r_{A'} = C(r_A)$, $a_{B'} = C(a_B)$ and $r_{B'} = C(r_B)$. Since $\alpha_1(A', B') = \alpha_1(A, B)$, we have

$$\begin{aligned} \frac{\langle \mathbf{x}_B, \mathbf{r}_A \rangle \langle \mathbf{a}_B, \mathbf{a}_A \rangle}{\langle \mathbf{a}_B, \mathbf{r}_A \rangle \langle \mathbf{x}_B, \mathbf{a}_A \rangle} &= \frac{\langle \mathbf{x}'_B, \mathbf{r}'_A \rangle \langle \mathbf{a}'_B, \mathbf{a}'_A \rangle}{\langle \mathbf{a}'_B, \mathbf{r}'_A \rangle \langle \mathbf{x}'_B, \mathbf{a}'_A \rangle} \\ &= \frac{\langle C^{-1}(\mathbf{x}'_B), \mathbf{r}_A \rangle \langle \mathbf{a}_B, \mathbf{a}_A \rangle}{\langle \mathbf{a}_B, \mathbf{r}_A \rangle \langle C^{-1}(\mathbf{x}'_B), \mathbf{a}_A \rangle} \\ \implies \frac{\langle \mathbf{x}_B, \mathbf{r}_A \rangle}{\langle C^{-1}(\mathbf{x}'_B), \mathbf{r}_A \rangle} &= \frac{\langle \mathbf{x}_B, \mathbf{a}_A \rangle}{\langle C^{-1}(\mathbf{x}'_B), \mathbf{a}_A \rangle} \end{aligned}$$

Let

$$\lambda = \frac{\langle \mathbf{x}_B, \mathbf{r}_A \rangle}{\langle C^{-1}(\mathbf{x}'_B), \mathbf{r}_A \rangle} = \frac{\langle \mathbf{x}_B, \mathbf{a}_A \rangle}{\langle C^{-1}(\mathbf{x}'_B), \mathbf{a}_A \rangle}$$

This implies

$$(5.5) \quad \langle \mathbf{x}_B - \lambda C^{-1}(\mathbf{x}'_B), \mathbf{r}_A \rangle = 0.$$

$$(5.6) \quad \langle \mathbf{x}_B - \lambda C^{-1}(\mathbf{x}'_B), \mathbf{a}_A \rangle = 0.$$

On the other hand, note that

$$(5.7) \quad \langle \mathbf{x}_B - \lambda C^{-1}(\mathbf{x}'_B), \mathbf{r}_B \rangle = \langle \mathbf{x}_B, \mathbf{r}_B \rangle - \bar{\lambda} \langle C^{-1}(\mathbf{x}'_B) - \mathbf{r}_B \rangle = 0 - \bar{\lambda} \langle \mathbf{x}'_B, \mathbf{r}'_B \rangle = 0.$$

Similarly,

$$(5.8) \quad \langle \mathbf{x}_B - \lambda C^{-1}(\mathbf{x}'_B), \mathbf{a}_B \rangle = 0.$$

Let L_A and L_B denote the two-dimensional time-like subspaces of $\mathbb{C}^{3,1}$ that represent the complex axes of A and B respectively. Thus $\{\mathbf{a}_A, \mathbf{r}_A\}$ and $\{\mathbf{a}_B, \mathbf{r}_B\}$ are the respective bases of L_A and L_B .

It follows from (5.5) – (5.8) that $v = \mathbf{x}_B - \lambda C^{-1}(\mathbf{x}'_B)$ is orthogonal to both L_A and L_B . We must have $\langle v, v \rangle > 0$. Thus v is polar to the \mathbb{C}^2 -chain (copy of $\mathbf{H}_{\mathbb{C}}^2$) that is represented by $V = v^\perp$. Since $\mathbb{C}^{3,1} = V \oplus \mathbb{C}v$, hence L_A and L_B must be subsets in V . Thus the fixed points of A and B belongs to boundary of the \mathbb{C}^2 -chain $\mathbb{P}(V)$. This is a contradiction to the non-singularity of (A, B) . Hence we must have $v = 0$, that is $C(\mathbf{x}_B) = \lambda \mathbf{x}'_B$. Thus, $C(x_B) = x'_B$. Consequently, $C(y_B) = y'_B$.

Similarly $\beta_1(A, B) = \beta_1(A', B')$ implies $C(x_A) = x'_A$ and hence $C(y_A) = y'_A$. This proves the lemma. \square

5.2.1. Proof of Theorem 1.1.

Proof. Suppose that A, B, A', B' are loxodromic elements such that

$$\text{tr}(A) = \text{tr}(A'), \text{tr}(B) = \text{tr}(B'), \sigma(A) = \sigma(A'), \sigma(B) = \sigma(B');$$

$$\alpha_i(A, B) = \alpha_i(A', B'), \beta_j(A, B) = \beta_j(A', B') \text{ and for } k = 1, 2, 3, \mathbb{X}_k(A, B) = \mathbb{X}_k(A', B').$$

Following the notation in Section 2.3, $A = C_A E_A C_A^{-1}$, $B = C_B E_B C_B^{-1}$ and similarly for A' and B' . Since the cross-ratios are equal, by Lemma 5.1 it follows that there exist $C \in \text{SU}(3, 1)$ such that $C(a_A) = a_{A'}, C(x_A) = x_{A'}, C(y_A) = y_{A'}, C(r_A) = r_{A'}$ and $C(a_B) = a_{B'}, C(x_B) = x_{B'}, C(y_B) = y_{B'}, C(r_B) = r_{B'}$. Therefore CAC^{-1} and A' have same eigenvectors. Since $\text{tr}(A') = \text{tr}(CAC^{-1})$, $\sigma(A') = \sigma(CAC^{-1})$, by Corollary 2.3 and Proposition 2.2, we must have $CAC^{-1} = A'$. Similarly, $B' = CBC^{-1}$. Thus $\langle A', B' \rangle = \langle CAC^{-1}, CBC^{-1} \rangle = C \langle A, B \rangle C^{-1}$ as claimed. \square

6. THE TWIST-BEND PARAMETER

Let $\langle A, B \rangle$ be a non-singular $(0, 3)$ group in $\text{SU}(3, 1)$, that is, A and B are loxodromic such that AB is also loxodromic. We want to attach two such non-singular subgroups to get a group that is freely generated by three generators. Now two cases are possible. The first case corresponds to the case when two different pairs of pants are attached along their boundary components. In this case, the $(0, 3)$ groups corresponds to different pairs of pants and they give a $(0, 4)$ group. The second case corresponds to the case when two of the boundary components of the same pair of pants is attached to give a torus. In this case attaching two $(0, 3)$ groups yields a $(1, 1)$ group, that is a group generated by two loxodromic elements and their commutator. This process is called ‘closing a handle’. To get more details about the geometric description of these terminologies and their interpretations in terms of group theoretic operations we refer to Parker-Platis [15].

Unless stated otherwise, the two-generator subgroups in this section are always assumed to be non-singular. Let $\langle A, B \rangle$ and $\langle C, D \rangle$ be two such $(0, 3)$ groups in $\text{SU}(3, 1)$ such that the boundary components associated to A and D are compatible, i.e. $A = D^{-1}$. A *complex hyperbolic twist bend* corresponds to an element K in $\text{SU}(3, 1)$ that commutes with A and conjugates $\langle C, D \rangle$, see

Parker-Platis [15, Section 8.1] for the ideas behind this notion. We assume that up to conjugacy, A fixes $0, \infty$ and of the form $E(\lambda, \phi)$ for some $(\lambda, \phi) \in S$. Since K commutes with A , it will also be of the form $K = E(\kappa, \psi)$ for some $(\kappa, \psi) \in S$, see [11]. Thus K will either be a boundary elliptic or loxodromic. The parameters (κ, ψ) obtained this way is the *twist-bend parameter*. It should be noted that the twist-bend is a relative invariant. It should always be chosen with respect to some fixed group $\langle A, B, C \rangle$ that one has to specify before applying the twist-bend. It gives us the degrees of freedom that is needed while attaching boundaries of pairs of pants to obtain a two-holed sphere that is fixed at the beginning. We say that the twist-bend parameter (κ, ψ) *oriented consistently* with A if when we write $A = QE(\lambda, \phi)Q^{-1}$, the matrix K is given by $QE(\kappa, \psi)Q^{-1}$.

In order to obtain conjugacy-invariant way to measure the twist-bend parameter, we define the following quantities:

$$\begin{aligned}\tilde{\mathbb{X}}_1(\kappa, \psi) &= [\mathbf{a}_B, \mathbf{a}_A, \mathbf{r}_A, K(\mathbf{r}_C)], \quad \tilde{\mathbb{X}}_2(\kappa, \psi) = [\mathbf{a}_B, \mathbf{r}_A, \mathbf{a}_A, K(\mathbf{r}_C)], \\ \tilde{\beta}_1(\kappa, \psi) &= [K(\mathbf{r}_C), \mathbf{a}_B, \mathbf{x}_A, \mathbf{a}_A], \quad \tilde{\beta}_2(\kappa, \psi) = [K(\mathbf{r}_C), \mathbf{a}_B, \mathbf{y}_A, \mathbf{a}_A].\end{aligned}$$

Lemma 6.1. *Let A, B, C be loxodromic elements in $\mathrm{SU}(3, 1)$ such that $\langle A, B \rangle$ and $\langle A^{-1}, C \rangle$ are non-singular. Let (κ, ψ) and (κ', ψ') be twist-bend parameters that are oriented consistently with A . If*

$$\tilde{\mathbb{X}}_1(\kappa, \psi) = \tilde{\mathbb{X}}_1(\kappa', \psi'), \quad \tilde{\mathbb{X}}_2(\kappa, \psi) = \tilde{\mathbb{X}}_2(\kappa', \psi'), \quad \text{and} \quad \tilde{\beta}_i(\kappa, \psi) = \tilde{\beta}_i(\kappa', \psi'), \quad i = 1 \text{ or } 2$$

then $\kappa = \kappa', \psi = \psi'$.

Proof. We shall prove the lemma assuming $\nu_1(A, C) \neq 0$. The other case is similar.

Without loss of generality, assume that A fixes 0 and ∞ and up to conjugacy $A = E(\lambda, \phi)$. So, we can further assume $K = E(\kappa, \psi)$. Let $a_A = \infty, r_A = 0$. Thus

$$\mathbf{a}_A = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{r}_A = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Let

$$\mathbf{a}_B = \begin{bmatrix} a \\ d \\ g \\ h \end{bmatrix}, \quad \mathbf{r}_B = \begin{bmatrix} c \\ f \\ j \\ t \end{bmatrix}, \quad \mathbf{r}_C = \begin{bmatrix} c' \\ f' \\ j' \\ t' \end{bmatrix}.$$

Further we assume without loss of generality, $\mathbf{x}_A = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$. Since $\mathbf{a}_A, \mathbf{r}_B, \mathbf{r}_C$ are light-like vectors,

a, h, t', c' are non-zero numbers and hence we have

$$(6.1) \quad \frac{\tilde{\mathbb{X}}_1(\kappa, \psi)}{\tilde{\mathbb{X}}_2(\kappa, \psi)} = \frac{\tilde{\mathbb{X}}_1(\kappa', \psi')}{\tilde{\mathbb{X}}_2(\kappa', \psi')}$$

$$(6.2) \quad \Rightarrow \frac{t'e^{-\bar{\kappa}}\bar{a}}{c'e^{\kappa}\bar{h}} = \frac{t'e^{-\bar{\kappa}'}\bar{a}}{c'e^{\kappa'}\bar{h}}$$

$$(6.3) \quad \Rightarrow \kappa = \kappa'.$$

Next $\tilde{\beta}_1(\kappa, \psi) = \tilde{\beta}_1(\kappa', \psi')$ implies

$$\frac{\bar{f}'e^{i\psi + \frac{\kappa - \bar{\kappa}}{2}}}{\bar{t}'e^{-\kappa}} = \frac{\bar{f}'e^{i\psi' + \frac{\kappa' - \bar{\kappa}'}{2}}}{\bar{t}'e^{-\kappa'}}.$$

As $\nu_1(A, C) \neq 0, f' \neq 0$ and since $\kappa = \kappa'$, we must have $\psi = \psi'$. □

6.1. Attaching two pairs of pants. A $(0, 4)$ subgroup of $SU(3, 1)$ is a group with four loxodromic generators such that their product is identity. These four loxodromic maps corresponds to the boundary curves of the four-holed spheres and are called *peripheral*. Thus a $(0, 4)$ group is freely generated by any of these three loxodromic elements.

Let $\langle A, B \rangle$ and $\langle C, D \rangle$ are two $(0, 3)$ groups with $A^{-1} = D$. Algebraically, a $(0, 4)$ group is constructed by the amalgamated free product of these groups with amalgamation along the common cyclic subgroup $\langle A \rangle$. Conjugating $\langle C, D \rangle$ by the twist-bend K yields a new $(0, 4)$ subgroup that is depended on K . We note the following lemme whose proof goes the same as Lemma 8.3 of Parker-Platis [15, p-131].

Lemma 6.2. *Suppose $\Gamma_1 = \langle A, B \rangle$ and $\Gamma_2 = \langle C, D \rangle$ are two $(0, 3)$ groups with peripheral elements $A, B, B^{-1}A^{-1}$ and $C, D, D^{-1}C^{-1}$ respectively. Moreover suppose that $A = D^{-1}$. Let K be any element of $SU(3, 1)$ that commutes with $A = D^{-1}$. Then the group $\langle A, B, KCK^{-1} \rangle$ is a $(0, 4)$ group with peripheral elements $B, B^{-1}A^{-1}, KCK^{-1}$ and $KD^{-1}C^{-1}K^{-1}$.*

Proposition 6.3. *Suppose that $\langle A, B \rangle$ and $\langle C, A^{-1} \rangle$ are two non-singular $(0, 3)$ groups. Let (κ, ψ) be a twist-bend parameter oriented consistently with A and let $\langle A, B, KCK^{-1} \rangle$ be the corresponding $(0, 4)$ group. Then $\langle A, B, KCK^{-1} \rangle$ is uniquely determined up to conjugation in $SU(3, 1)$ by the Fenchel-Nielsen coordinates:*

$tr(A), tr(B), tr(C), \sigma(A), \sigma(B), \sigma(C), \mathbb{X}_k(A, B), \mathbb{X}_k(A, C), k = 1, 2, 3$, two non-zero α -invariants: $\alpha_i(A, B), \alpha_l(A, C)$, two non-zero β -invariants: $\beta_j(A, B), \beta_m(A, C)$ and the twist-bend parameter (κ, ψ) .

In the parameter space associated to $\langle A, B, KCK^{-1} \rangle$, the parameters corresponding to traces, α and β are complex numbers, the elements $(\mathbb{X}_1, \mathbb{X}_2, \mathbb{X}_3)$ belongs to the cross-ratio variety that is 5 real dimensional and (κ, ψ) has real dimension three. Thus we need a total of 30 real parameters to specify $\langle A, B, KCK^{-1} \rangle$ up to conjugacy.

The number 30 is obtained by the following count:

$$[(3 \text{ (traces)} \times 2) + (3 \text{ (}\sigma\text{-invariants)} \times 1) + (2 \text{ (cross-ratios)} \times 5) + (2 \text{ (}\alpha\text{-invariants)} \times 2) + (2 \text{ (}\beta\text{-invariants)} \times 2) + 3 \text{ one twist-bend} = 30].$$

Proof. Suppose $\langle A, B, KCK^{-1} \rangle$ and $\langle A', B', K'C'K'^{-1} \rangle$ are two $(0, 4)$ subgroups having the same Fenchel-Nielsen coordinates. Suppose, we have

$$(6.4) \quad tr(A) = tr(A'), \quad tr(B) = tr(B'), \quad tr(C) = tr(C');$$

$$(6.5) \quad \sigma(A) = \sigma(A'), \quad \sigma(B) = \sigma(B'), \quad \sigma(C) = \sigma(C');$$

$$(6.6) \quad \text{for } t = 1, 2, 3, \quad \mathbb{X}_t(A, B) = \mathbb{X}_t(A', B'), \quad \mathbb{X}_t(A, C) = \mathbb{X}_t(A', C');$$

$$(6.7) \quad \alpha_i(A, B) = \alpha_i(A', B'), \quad \alpha_l(A, C) = \alpha_l(A', C'),$$

$$(6.8) \quad \beta_j(A, B) = \beta_j(A', B'), \quad \beta_m(A, C) = \beta_m(A', C');$$

$$(6.9) \quad (\kappa, \psi) = (\kappa', \psi').$$

Using these relations, it follows from Theorem 1.1 that there exists C_1 and C_2 in $SU(3, 1)$ that conjugates $\langle A, B \rangle$ and $\langle C, A^{-1} \rangle$ respectively to $\langle A', B' \rangle$ and $\langle C', A'^{-1} \rangle$.

Now the twist-bends are defined with respect to the same initial group $\langle A, B, C \rangle$ that we fix at the beginning before defining the twist-bend and attaching the two $(0, 3)$ groups. So, without loss of generality, we may assume $A = A', B = B', C = C'$ and thus $C_1 = C_2$. Now with respect to the same initial group $\langle A, B, C \rangle$, by (6.9) it follows that $K = K'$. This implies that $\langle A, B, KCK^{-1} \rangle$ is determined uniquely up to conjugacy.

Conversely, suppose that $\langle A, B, KCK^{-1} \rangle$ and $\langle A', B', K'C'K'^{-1} \rangle$ are conjugate. Then clearly, (6.4) – (6.8) are satisfied. The only thing remains to show is (6.9). Now by the invariance of the cross-ratios it follows that

$$\tilde{X}_1(\kappa, \psi) = \tilde{X}_1(\kappa', \psi'), \quad \tilde{X}_2(\kappa, \psi) = \tilde{X}_2(\kappa', \psi'), \quad \tilde{\beta}_k(\kappa, \psi) = \tilde{\beta}_k(\kappa', \psi'),$$

and hence by Lemma 6.1, $(\kappa, \psi) = (\kappa', \psi')$. \square

6.2. Closing a handle. We are now interested in obtaining a one-holed torus by attaching two holes of the same pair of pants in the complex hyperbolic 3-space. The process of attaching these two holes is called *closing a handle*. Geometrically, it corresponds to attach two boundary components of the same pair of pants. To make this work, one of the peripheral elements of the corresponding $(0, 3)$ group must be conjugate to the inverse of the other peripheral element. This ensures that they are compatible for the attachment. Suppose the two peripheral elements are A and $BA^{-1}B^{-1}$, then the third element would be $[A, B] = BAB^{-1}A^{-1}$. A $(1, 1)$ subgroup of $SU(3, 1)$ is a group that is generated by the elements A , B and $[A, B]$. From group theoretic viewpoint, closing a handle is same as taking the HNN-extension of the $(0, 3)$ group $\langle A, BA^{-1}B^{-1} \rangle$ by adjoining the element B to form a $(1, 1)$ group. When we take the HNN-extension, the map B is not unique. If K is any element in $SU(3, 1)$ that commutes with A , then $\langle A, BK \rangle$ gives another $(1, 1)$ group. Varying K corresponds to a twist-bend coordinate as above.

If $A = QE(\lambda, \phi)Q^{-1}$ for $(\lambda, \phi) \in S$, we define the twist-bend parameter by (κ, ψ) by $K = QE(\kappa, \psi)Q^{-1}$ just as before and we say (κ, ψ) is oriented consistently with A . In this case also (κ, ψ) is defined relative to a reference group that we fix at the starting of the attachment.

Lemma 6.4. *Let $\langle A, BA^{-1}B^{-1} \rangle$ be a non-singular $(0, 3)$ group. Let B be a fixed choice of an element in $SU(3, 1)$ conjugating A^{-1} to $BA^{-1}B^{-1}$. Let (κ, ψ) (κ', ψ') be twist-bend parameters oriented consistently with A . Then $\langle A, BK \rangle$ is conjugate to $\langle A, BK' \rangle$ if and only if $(\kappa, \psi) = (\kappa', \psi')$.*

Proof. If $(\kappa, \psi) = (\kappa', \psi')$, then clearly $K = K'$ and hence the groups are equal.

Conversely, suppose $\langle A, BK \rangle$ is conjugate to $\langle A, BK' \rangle$. Then conjugating element D must commute with A . Hence $D(a_A) = a_A$, $D(r_A) = r_A$. Since $BA^{-1}B^{-1}$ has been fixed at the beginning, we have

$$BA^{-1}B^{-1} = (BK')^{-1}A^{-1}(BK') = (DBKD^{-1})A^{-1}(BBKD^{-1})^{-1} = D(BA^{-1}B^{-1})D^{-1}.$$

Thus D commutes with $BA^{-1}B^{-1}$ and fixes $a_{BA^{-1}B^{-1}} = B(a_A)$, $r_{BA^{-1}B^{-1}} = B(r_A)$. Since the fixed points are distinct, D is either identity or the fixed points belong to the same chain fixed by D . But the later is not possible by non-singularity of the $(0, 3)$ group. Hence D must be identity. Thus $BK' = BK$ and hence $K = K'$, i.e $(\kappa, \psi) = (\kappa', \psi')$. \square

Proposition 6.5. *Let $\langle A, BK \rangle$ be an $(1, 1)$ group obtained from the non-singular $(0, 3)$ group $\langle A, BA^{-1}B^{-1} \rangle$ by closing a handle with associated twist-bend parameter (κ, ψ) . Then $\langle A, BK \rangle$ is determined uniquely up to conjugation by its Fenchel-Nielsen coordinates*

$tr(A)$, $\sigma(A)$, $\mathbb{X}_k(A, BA^{-1}B^{-1})$, $k = 1, 2, 3$, one non-zero α -invariant: $\alpha_i(A, BA^{-1}B^{-1})$, one non-zero β -invariant: $\beta_j(A, BA^{-1}B^{-1})$ and the twist-bend parameter (κ, ψ) .

Thus, we need 15 real parameters to specify $\langle A, BK \rangle$ up to conjugacy.

Proof. Suppose that $\langle A, BK \rangle$ and $\langle A, B'K' \rangle$ are two $(1, 1)$ groups with the same Fenchel-Nielsen coordinates. In particular $tr(A) = tr(A')$ and hence

$$tr(BA^{-1}B^{-1}) = \overline{tr(A)} = \overline{tr(A')} = tr(B'A'^{-1}B'^{-1}).$$

Further using $X_k(A, BA^{-1}B^{-1}) = \mathbb{X}_k(A', B'A'^{-1}B'^{-1})$ for $k = 1, 2, 3$,

$\alpha_i(A, BA^{-1}B^{-1}) = \alpha_i(A', B'A'^{-1}B'^{-1})$ and $\beta_j(A, BA^{-1}B^{-1}) = \beta_j(A', B'A'^{-1}B'^{-1})$, we see by Theorem 1.1 that the $(0, 3)$ groups $\langle A, BA^{-1}B^{-1} \rangle$ and $\langle A', B'A'^{-1}B'^{-1} \rangle$ are conjugate. Thus we can assume $A = A'$, $BA^{-1}B^{-1} = B'A'^{-1}B'^{-1}$. Now using the above lemma we see that $(\kappa, \psi) = (\kappa', \psi')$. Hence $K = K'$. Thus the group $\langle A, BK \rangle$ is determined uniquely up to conjugation.

Conversely, suppose $\langle A, BK \rangle$ and $\langle A', B'K' \rangle$ are conjugate. Hence it is clear that all the Fenchel-Nielsen coordinates but the twist-bend parameters are the same. Conjugating, if necessary, we assume $A = A'$. Since B is a fixed choice of conjugation element with reference to which (κ, ψ) and (κ', ψ') are defined, we may also assume $B = B'$. Now using Lemma 6.4 we see that $K = K'$, i.e. $(\kappa, \psi) = (\kappa', \psi')$. \square

7. PROOF OF THEOREM 1.3

We have seen in Theorem 1.1 that each $(0, 3)$ group $\langle A, B \rangle$ is determined up to conjugacy by 15 real parameters. Each pair of pants in the pant decomposition (complement of \mathcal{C}) corresponds to a $(0, 3)$ group. While we attach two pairs of pants, we attach two $(0, 3)$ groups subject to the compatibility condition that one peripheral element in a group is conjugate to the inverse of a peripheral element in the other group. Thus we get a $(0, 4)$ group that is specified up to conjugacy by 30 real parameters described in Proposition 6.3. Continuing this way, when we attached $2g - 2$ of our $(0, 3)$ groups, we need a total $15(2g - 2) = 30g - 30$ real parameters to specify the resulting surface with $2g$ handles. These handles corresponds to the g curves that correspond to two boundary components of the same three-holed sphere. Now there are g complex constraints that is imposed to close these handles: one of the peripheral elements of each of these $(0, 3)$ groups, must be conjugate to the inverse of the other peripheral element. And to each peripheral element there are 3 natural real parameters: the trace and the σ -invariant. So, the number of real parameters now reduced to $30g - 30 - 3g = 27g - 30$. But there are g twist-bend parameters (κ_i, ψ_i) one for each handle and each contributes 3 real dimensions. Thus we need a total of $27g - 30 + 3g = 30g - 30$ real parameters to specify ρ up to conjugacy.

If two representations have the same coordinates, then the coordinates of the $(0, 3)$ groups are the same, so they are conjugate. Further it follows from Proposition 6.3 and Proposition 6.5 that the $(0, 4)$ groups and $(1, 1)$ groups are also determined uniquely up to conjugacy while attaching the $(0, 3)$ groups. Hence the representations having the same parameters are conjugate. Conversely, if two representations are conjugate, then clearly they will have the same coordinates.

This proves the theorem.

7.1. Remarks. Let $\langle A, B \rangle$ be a non-singular $(0, 3)$ group whose peripheral elements are represented by A, B and $B^{-1}A^{-1}$ respectively. Let $A = E(\lambda, \phi)$ and let $B = C_B E(\mu, \psi) C_B^{-1}$, where

$$C_B = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ j & k & l & m \\ n & s & p & q \end{bmatrix}, \quad C_B^{-1} = \begin{bmatrix} \bar{q} & \bar{h} & \bar{m} & \bar{d} \\ \bar{s} & \bar{f} & \bar{k} & \bar{b} \\ \bar{p} & \bar{g} & \bar{l} & \bar{c} \\ \bar{n} & \bar{e} & \bar{j} & \bar{a} \end{bmatrix}.$$

Thus

$$B = \begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ e_1 & f_1 & g_1 & h_1 \\ j_1 & k_1 & l_1 & m_1 \\ n_1 & s_1 & p_1 & q_1 \end{bmatrix}, \quad \text{where}$$

$$\begin{aligned} a_1 &= e^\mu a \bar{q} + e^{\frac{\bar{\mu}-\mu}{2}-i\psi} b \bar{s} + e^{\frac{\bar{\mu}-\mu}{2}+i\psi} c \bar{p} + e^{-\bar{\mu}} d \bar{n}, \\ b_1 &= e^\mu a \bar{h} + e^{\frac{\bar{\mu}-\mu}{2}-i\psi} b \bar{f} + e^{\frac{\bar{\mu}-\mu}{2}+i\psi} c \bar{g} + e^{-\bar{\mu}} d \bar{e}, \\ c_1 &= e^\mu a \bar{m} + e^{\frac{\bar{\mu}-\mu}{2}-i\psi} b \bar{k} + e^{\frac{\bar{\mu}-\mu}{2}+i\psi} c \bar{l} + e^{-\bar{\mu}} d \bar{j}, \\ d_1 &= e^\mu a \bar{d} + e^{\frac{\bar{\mu}-\mu}{2}-i\psi} |b|^2 + e^{\frac{\bar{\mu}-\mu}{2}+i\psi} |c|^2 + e^{-\bar{\mu}} d \bar{a}, \\ e_1 &= e^\mu e \bar{q} + e^{\frac{\bar{\mu}-\mu}{2}-i\psi} f \bar{s} + e^{\frac{\bar{\mu}-\mu}{2}+i\psi} g \bar{p} + e^{-\bar{\mu}} h \bar{n}, \\ f_1 &= e^\mu e \bar{h} + e^{\frac{\bar{\mu}-\mu}{2}-i\psi} |f|^2 + e^{\frac{\bar{\mu}-\mu}{2}+i\psi} |g|^2 + e^{-\bar{\mu}} h \bar{e}, \\ g_1 &= e^\mu e \bar{m} + e^{\frac{\bar{\mu}-\mu}{2}-i\psi} f \bar{k} + e^{\frac{\bar{\mu}-\mu}{2}+i\psi} g \bar{l} + e^{-\bar{\mu}} h \bar{j}, \\ h_1 &= e^\mu e \bar{d} + e^{\frac{\bar{\mu}-\mu}{2}-i\psi} f \bar{b} + e^{\frac{\bar{\mu}-\mu}{2}+i\psi} g \bar{c} + e^{-\bar{\mu}} h \bar{a}, \end{aligned}$$

$$\begin{aligned}
j_1 &= e^\mu j \bar{g} + e^{\frac{\bar{\mu}-\mu}{2}-i\psi} k \bar{s} + e^{\frac{\bar{\mu}-\mu}{2}+i\psi} l \bar{p} + e^{-\bar{\mu}} m \bar{n}, \\
k_1 &= e^\mu j \bar{h} + e^{\frac{\bar{\mu}-\mu}{2}-i\psi} k \bar{f} + e^{\frac{\bar{\mu}-\mu}{2}+i\psi} l \bar{g} + e^{-\bar{\mu}} m \bar{e}, \\
l_1 &= e^\mu j \bar{m} + e^{\frac{\bar{\mu}-\mu}{2}-i\psi} |k|^2 + e^{\frac{\bar{\mu}-\mu}{2}+i\psi} |l|^2 + e^{-\bar{\mu}} b \bar{j}, \\
m_1 &= e^\mu j \bar{d} + e^{\frac{\bar{\mu}-\mu}{2}-i\psi} k \bar{b} + e^{\frac{\bar{\mu}-\mu}{2}+i\psi} l \bar{c} + e^{-\bar{\mu}} m \bar{a}, \\
n_1 &= e^\mu n \bar{q} + e^{\frac{\bar{\mu}-\mu}{2}-i\psi} |s|^2 + e^{\frac{\bar{\mu}-\mu}{2}+i\psi} |p|^2 + e^{-\bar{\mu}} q \bar{n}, \\
s_1 &= e^\mu n \bar{h} + e^{\frac{\bar{\mu}-\mu}{2}-i\psi} s \bar{f} + e^{\frac{\bar{\mu}-\mu}{2}+i\psi} p \bar{g} + e^{-\bar{\mu}} q \bar{e}, \\
p_1 &= e^\mu n \bar{m} + e^{\frac{\bar{\mu}-\mu}{2}-i\psi} s \bar{k} + e^{\frac{\bar{\mu}-\mu}{2}+i\psi} p \bar{l} + e^{-\bar{\mu}} q \bar{j}, \\
q_1 &= e^\mu n \bar{d} + e^{\frac{\bar{\mu}-\mu}{2}-i\psi} s \bar{b} + e^{\frac{\bar{\mu}-\mu}{2}+i\psi} p \bar{c} + e^{-\bar{\mu}} q \bar{a}
\end{aligned}$$

Hence

$$B^{-1}A^{-1} = \begin{bmatrix} e^{-\lambda} \bar{q}_1 & e^{\frac{\lambda-\bar{\lambda}}{2}+i\phi} \bar{h}_1 & e^{\frac{\lambda-\bar{\lambda}}{2}-i\phi} \bar{m}_1 & e^{\bar{\lambda}} \bar{d}_1 \\ e^{-\lambda} \bar{s}_1 & e^{\frac{\lambda-\bar{\lambda}}{2}+i\phi} \bar{f}_1 & e^{\frac{\lambda-\bar{\lambda}}{2}-i\phi} \bar{k}_1 & e^{\bar{\lambda}} \bar{b}_1 \\ e^{-\lambda} \bar{p}_1 & e^{\frac{\lambda-\bar{\lambda}}{2}+i\phi} \bar{g}_1 & e^{\frac{\lambda-\bar{\lambda}}{2}-i\phi} \bar{l}_1 & e^{\bar{\lambda}} \bar{c}_1 \\ e^{-\lambda} \bar{n}_1 & e^{\frac{\lambda-\bar{\lambda}}{2}+i\phi} \bar{e}_1 & e^{\frac{\lambda-\bar{\lambda}}{2}-i\phi} \bar{j}_1 & e^{\bar{\lambda}} \bar{a}_1 \end{bmatrix}.$$

$$\mathbb{X}_1(A, B) = \bar{a}q, \quad \mathbb{X}_2(A, B) = \bar{n}d, \quad \mathbb{X}_3(A, B) = \frac{nd}{aq}.$$

Note that after choosing a suitable lift such that $\langle \mathbf{a}_B, \mathbf{r}_B \rangle = 1$, we see that

$$\eta_1(A, B) = \bar{s}b, \quad \eta_2(A, B) = \bar{p}c, \quad \nu_1(A, B) = e\bar{h}, \quad \nu_2(A, B) = j\bar{m},$$

And these quantities are appearing naturally in the expressions of B and $B^{-1}A^{-1}$. Further if one of the η 's and one of the ν 's are non-zero, then this also implies the existence of the α and β -invariants. Also it follows that there will be real analytic change of the trace parameters under a choice of peripheral elements of a $(0, 3)$ group. Let us do the computations for $\text{tr}(B^{-1}A^{-1})$. We have the relations: $1 = b\bar{s} + |f|^2 + |k|^2 + s\bar{b} = c\bar{p} + |g|^2 + |l|^2 + p\bar{c} = h\bar{e} + |f|^2 + |g|^2 + e\bar{h} = m\bar{j} + |k|^2 + |l|^2 + j\bar{m}$. Solving these four equations, we have $|f|, |g|, |k|$ and $|l|$ are real analytic functions of η_1, η_2, ν_1 and ν_2 . Therefore a_1, f_1, l_1 and q_1 are real analytic functions of $\mu, \psi, \eta_1, \eta_2, \nu_1, \nu_2$. Now we observe the following lemma:

Lemma 7.1. *Let A, B be two loxodromic elements in $\text{SU}(3, 1)$. Then the following relations hold:*

$$(1) \quad \eta_1 \bar{\alpha}_1 + \eta_2 \bar{\alpha}_2 = -(\mathbb{X}_2 + \bar{\mathbb{X}}_3 \bar{\mathbb{X}}_1).$$

$$(2) \quad \nu_1 \bar{\beta}_1 + \nu_2 \bar{\beta}_2 = -(\bar{\mathbb{X}}_2 + \mathbb{X}_3 \bar{\mathbb{X}}_1).$$

$$(3) \quad \frac{\eta_1}{\alpha_1} + \frac{\eta_2}{\alpha_2} = -(\bar{\mathbb{X}}_1 + \mathbb{X}_2 / \mathbb{X}_3).$$

$$(4) \quad \frac{\nu_1}{\beta_1} + \frac{\nu_2}{\beta_2} = -(\bar{\mathbb{X}}_1 + \bar{\mathbb{X}}_2 / \bar{\mathbb{X}}_3).$$

Proof. By symmetry, it is enough to prove (1). Take A, B as in previous section. Then we have

$$\eta_1(A, B) = \bar{s}b, \quad \eta_2(A, B) = \bar{p}c, \quad \alpha_1(A, B) = \frac{bn}{sa}, \quad \alpha_2(A, B) = \frac{cn}{pa},$$

$$\mathbb{X}_1(A, B) = \bar{a}q, \quad \mathbb{X}_2(A, B) = \bar{n}d, \quad \mathbb{X}_3(A, B) = \frac{nd}{aq}.$$

Observe that $\eta_1 \bar{\alpha}_1 + \eta_2 \bar{\alpha}_2 = -\frac{\bar{n}}{a}(b\bar{b} + c\bar{c})$. Then $d\bar{a} + b\bar{b} + c\bar{c} + a\bar{d} = 0$ and $(a\bar{n})(q\bar{d}) = |\mathbb{X}_1|^2 \bar{\mathbb{X}}_3$ implies $\eta_1 \bar{\alpha}_1 + \eta_2 \bar{\alpha}_2 = -(\mathbb{X}_2 + \bar{\mathbb{X}}_3 \bar{\mathbb{X}}_1)$ and the result follows. \square

This lemma suggests that $\alpha_1, \alpha_2, \beta_1, \beta_2$ are real differentiable functions of $\eta_1, \eta_2, \nu_1, \nu_2$ and \mathbb{X}_k for $k = 1, 2, 3$.

With the above observations, we have the feeling that the following conjecture might be true in general.

Conjecture 1. *Let $\langle A, B \rangle$ be a Zariski-dense free subgroup of $SU(3, 1)$ generated by loxodromic elements A and B . Then $\langle A, B \rangle$ is determined up to conjugacy by the following parameters:*

$$tr(A), tr(B), \sigma(A), \sigma(B), \mathbb{X}_k(A, B), k = 1, 2, 3, \text{ one } \eta\text{-invariant and one } \nu\text{-invariant,}$$

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